# MATH 350 Linear Algebra
# Quiz 9 Solutions

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## 1 Solutions

Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by the matrix

$$A = \begin{bmatrix} 8 & -4 & -8 \\ -15 & 16 & 22 \\ 12 & -10 & -15 \end{bmatrix}$$

This matrix is similar to

$$A^c := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and hence has characteristic polynomial

$$\chi_A = \chi_{A^c} = (2-t)^2(5-t)$$

Thus, it has eigenvalues 2 and 5, with multiplicity 2 and 1 respectively. Let $g_2(t), g_5(t)$ be polynomials such that

$$g_2(t)(2-t)^2 + g_5(t)(5-t) = 1$$

(Such polynomials exist since $(2-t)^2, (5-t)$ are relatively prime)

(a) Let $v \in \mathbb{R}^3$ be arbitrary. Define $v_5 := g_2(A)(2-A)^2 v$ and $v_2 := g_5(A)(5-A)v$, so that $v = v_2 + v_5$. Show that $v_2 \in K_2, v_5 \in K_5$.

*Proof.* We showed in class that $K_\lambda$ is the null space of $(A - \lambda I)^{m_\lambda}$, where $m_\lambda$ denotes the multiplicity of the eigenvalue $\lambda$. This is also Theorem 7.2 in the book. Thus, we want to show that $(A-5I)v_5 = (A-2I)v_2 = 0$. By definition of $v_5$, we have

$$\begin{aligned} (A-5I)v_5 &= (A-5I)g_2(A)(2-A)^2 v && \text{by definition} \\ &= g_2(A)(A-5I)(2-A)^2 v && \text{since polynomials in } A \text{ commute with each other} \\ &= -g_2(A)\chi_A(A)v && \text{by the definition of the characteristic polynomial} \\ &= -g_2(A)(\chi_A(A)(v)) && \text{In general if } S, T \text{ are linear operators then } STv := (ST)(v) = S(T(v)) \\ &= 0 && \text{by the Cayley Hamilton theorem} \end{aligned}$$

The case of $v_2$ is exactly analogous. □

(b) Show that if $v_2' \in K_2, v_5' \in K_5$ are other vectors satisfying $v = v_2' + v_5'$, then $v_2' = v_2$ and $v_5' = v_5$.

*Proof.* We know $v_2 + v_5 = v = v_2' + v_5'$. Thus, we have

$$v_2' - v_2 = v_5 - v_5'$$

The left hand side lies in $K_2$, and the right hand side lies in $K_5$, so both the left hand side and the right hand side lie in $K_2 \cap K_5$. Since $K_2 \cap K_5 = 0$ (we showed this in class; this is Theorem 7.1 in the book), this implies that both the left and right hand sides are 0, so $v_2' = v_2$ and $v_5' = v_5$.

Essentially the same problem was given on Homework 2 (§1.3, Problem 30). □