MATH 350 Linear Algebra
Quiz 8 Solutions

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1 Solutions

Let $T : V \to V$ be linear, $\dim V < \infty$, and $v \in V$ nonzero. Let k be the minimum positive integer such that $T^k(v)$ is a linear combination of the vectors in

$$\beta := \{v, Tv, \dots, T^{k-1}v\}$$

In other words, $T^k v = a_0 v + a_1 Tv + a_2 T^v + \cdots + a_{k-1}T^{k-1}v$ and for any positive integer k' less than k, $T^{k'}v$ is not a linear combination of $v, Tv, \dots, T^{k'-1}v$.

Recall that $\langle v \rangle_T$ was defined to be $\mathrm{Span}\{v, Tv, T^2v, \dots\}$.

(a) Show that β spans $\langle v \rangle_T$.

We want to show that $\mathrm{Span}\{v, Tv, \dots, T^{k-1}v\} = \mathrm{Span}\{v, Tv, T^2v, \dots\}$. Clearly the left side is contained in the right side (since the left side is the span of a subset). It remains to show that for every $n \geq 0$, $T^n v \in \mathrm{Span}\{v, Tv, \dots, T^{k-1}v\}$.

Idea. Here is an illustration of the idea, in the special case $k = 3$, so that $T^3v = v + Tv + T^2v$. We want to show that for all $n \geq 0$, $T^n v \in \mathrm{Span}\{v, Tv, T^2v\}$. The cases $n = 0, 1, 2$ are obvious. For $n = 3$, $T^3v = v + Tv + T^2v$ so T^3v lies in $\mathrm{Span}\{v, Tv, T^2v\}$. For $n = 4$, $T^4v = T(T^3v) = T(v + Tv + T^2v) = Tv + T^2v + T^3v = Tv + T^2v + (v + Tv + T^2v)$, so T^4v also lies in $\mathrm{Span}\{v, Tv, T^2v\}$. In general, if $x \in \mathrm{Span}\{v, Tv, T^2v\}$, writing $x = a_0v + a_1Tv + a_2T^2v$, then $Tx = a_0Tv + a_1T^2v + a_2T^3v$ which also lies in $\mathrm{Span}\{v, Tv, T^2v\}$ since Tv, T^2v, T^3v all lie in $\mathrm{Span}\{v, Tv, T^2v\}$. We can turn this into a rigorous proof using induction:

Proof. We will use induction on n. By assumption, $v, Tv, T^2v, \dots T^kv$ all lie in $\mathrm{Span}\{v, Tv, \dots, T^{k-1}v\}$. This will form our base case. Now if $T^nv \in \mathrm{Span}\{v, Tv, \dots, T^{k-1}v\}$, then we can write $T^nv = b_0v + b_1Tv + \cdots + b_{k-1}T^{k-1}v$, so

$$T^{n+1}v = T(T^nv) = T(b_0v + b_1Tv + \cdots + b_{k-1}T^{k-1}v) = b_0Tv + b_1T^2v + \cdots + b_{k-1}T^kv$$

Since $Tv, T^2v, \dots, T^kv$ all lie in $\mathrm{Span}\{v, Tv, \dots, T^{k-1}v\}$, it follows that $T^{n+1}v$ does as well. □

(b) Assuming that $\dim V = 5$, express the matrix $[T_{\langle v \rangle_T}]_\beta$ in terms of $a_0, a_1, \dots, a_{k-1}$. You may use the fact that β is a basis of $\langle v \rangle_T$.

Solution. Since β is a basis and $\dim V = 5$, we are in the situation $k = 5$. Thus $\beta = \{v, Tv, T^2v, T^3v, T^4v\}$, and $T^5v = a_0v + a_1Tv + a_2T^2v + a_3T^3v + a_4T^4v$. The ith basis vector β_i is $T^{i-1}v$, and the ith column of $[T_{\langle v \rangle_T}]_\beta$ is $[T(\beta_i)]_\beta$. Thus

- the first column is $[T(\beta_1)]_\beta = [Tv]_\beta = e_2$,
- the second column is $[T(\beta_2)]_\beta = [TTv]_\beta = [T^2v]_\beta = e_3$,
- the third column is $[T(\beta_3)]_\beta = [TT^2v]_\beta = [T^3v]_\beta = e_4$, and

- the fourth column is $[T(\beta_4)]_\beta = [TT^3v]_\beta = [T^4v]_\beta = e_5$.
- The fifth column is $[T(\beta_5)]_\beta = [TT^4v]_\beta = [T^5v]_\beta = [a_0v + a_1Tv + a_2T^2v + a_3T^3v + a_4T^4v]_\beta = (a_0, a_1, a_2, a_3, a_4)$.

Thus, the matrix is

$$[T_{\langle v \rangle_T}]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & a_0 \\ 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & a_4 \end{bmatrix}$$