MATH 350 Linear Algebra
Quiz 7 Solutions

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1 Solutions

(a) Let $T : V \to W$ be a linear map, with V, W finite dimensional. Let α, β be bases of V and W respectively. Define $[T]_\alpha^\beta$. If you want to use the word "coordinate vector", or the notation $[v]_\beta$, then you should define it.

Solution. Let $\alpha_1, \ldots, \alpha_n$ be the vectors in α and $\beta_1, \ldots, \beta_m$ the vectors in β. The ith column of the matrix $[T]_\alpha^\beta$ is the coordinate vector $[T(\alpha_i)]_\beta$. This means that for each i between 1 and n, the ith column is $(c_1, c_2, \ldots, c_m)$, where the $c_j \in F$ are the unique elements such that

$$T(\alpha_i) = c_1\beta_1 + c_2\beta_2 + \cdots + c_m\beta_m$$

(b) Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

Find an invertible matrix Q and a diagonal matrix D such that

$$Q^{-1}AQ = D$$

Solution. Since we want $Q^{-1}AQ$ to be diagonal, that is the same as asking for the property that there exist numbers $\lambda_1, \lambda_2, \lambda_3 \in F$ such that

$$Q^{-1}AQe_i = \lambda_i e_i \qquad \text{for } i = 1, 2, 3$$

Note that if Qe_i is an eigenvector of A with eigenvalue λ_i, then we have

$$Q^{-1}AQe_i = Q^{-1}A(Qe_i) = Q^{-1}\lambda_i Qe_i = \lambda_i Q^{-1}Qe_i = \lambda_i e_i$$

Thus we want Q to be an invertible matrix such that Qe_i is an eigenvector of A for each i. Since Qe_i is just the ith column of Q, once we have an eigenbasis of A, it is easy to read off the matrix Q. Thus, our task is to find an eigenbasis. Since eigenvalues are the roots of the characteristic polynomial χ_A, we begin by computing χ_A.

$$\chi_A(t) = \det(A - tI) = \det \begin{bmatrix} 1-t & 0 & -2 \\ -2 & 2-t & -3 \\ 0 & 0 & 3-t \end{bmatrix} = (3-t)(2-t)(1-t)$$

It follows that the eigenvalues of $1, 2, 3$, each having multiplicity 1. Moreover, the characteristic polynomial is split, and since dimensions of eigenspaces are always sandwiched between 1 and the multiplicity, the dimension of each eigenspace is equal to the multiplicty, so A is diagonalizable. This is just a sanity check to make sure we're not on a fool's errand.

To find an eigenbasis, we must find a basis for each eigenspace. For the eigenvalue 1, the eigenspace is

$$E_1 = N(A - I) = N\left(\begin{bmatrix} 0 & 0 & -2 \\ -2 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

It follows that $v_1 = (1, 2, 0)$ is an eigenbasis for the eigenvalue 1. Next,

$$E_2 = N(A - 2I) = N\left(\begin{bmatrix} -1 & 0 & -2 \\ -2 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

so $v_2 = (0, 1, 0)$ is an eigenbasis for the eigenvalue 2. Finally,

$$E_3 = N(A - 3I) = N\left(\begin{bmatrix} -2 & 0 & -2 \\ -2 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$$

so $v_3 = (1, 1, -1)$ is an eigenbasis for the eigenvalue 3. It follows that we can take Q to be the matrix $[v_1 \; v_2 \; v_3]$. In other words,

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

In which case we will have

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note: We could have also taken Q to be the matrix $[v_2 \; v_3 \; v_1]$ (or any other permutation of the columns). We could have also scaled each column of the matrix by any nonzero vector. For example, if

$$Q' := [2v_2 \quad v_3 \quad -v_1],$$

then we would have

$$(Q')^{-1}AQ' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$