MATH 350 Linear Algebra
Quiz 6 Solutions

Instructor: Will Chen

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1 Solutions

(a) Let V be an n-dimensional vector space, and let $I : V \to V$ be the identity linear transformation. Let β, γ be two bases for V, and let I_n denote the $n \times n$ identity matrix. Show that $[I]_\beta^\gamma = I_n$ if and only if $\beta = \gamma$.

Proof. By definition, the ith column of $[I]_\beta^\gamma$ is the coordinate vector $[I(\beta_i)]_\gamma = [\beta_i]_\gamma$. Recall that the ith column of the identity matrix I_n is just the column vector e_i (this has a 1 in the ith position and zeroes elsewhere). Thus we find that

$$(i\text{th column of } [I]_\beta^\gamma) = (i\text{th column of } I_n) \qquad \text{if and only if} \qquad [\beta_i]_\gamma = e_i$$

Recall the definition of the coordinate vector: for any $v \in V$, $[v]_\gamma$ is the column vector whose entries $a_1, \ldots, a_n$ satisfy

$$v = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$$

Thus, $[\beta_i]_\gamma$ is the column vector whose entries $a_1, \ldots, a_n$ satisfy

$$\beta_i = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$$

It follows that $[\beta_i]_\gamma = e_i$ if and only if

$$\beta_i = 0\gamma_1 + \cdots + 0\gamma_{i-1} + 1\gamma_i + 0\gamma_{i+1} + \cdots + 0\gamma_n = \gamma_i$$

Since $[I]_\beta^\gamma = I_n$ if and only if each column of $[I]_\beta^\gamma$ is equal to the corresponding column of I_n, we find that $[I]_\beta^\gamma = I_n$ if and only if $\beta_i = \gamma_i$ for each i.

$\square$

(b) Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity linear transformation. Show that every $n \times n$ invertible matrix can be written as $[I]_\beta^\gamma$ for suitable bases β, γ of $\mathbb{R}^n$. Hint: It suffices to take $\gamma = \text{std}$.

Proof. Let A be an arbitrary $n \times n$ invertible matrix. We'd like to show that we can find a basis β for $\mathbb{R}^n$ such that $A = [I]_\beta^{\text{std}}$. Since A is invertible, A has rank n, so its columns span $\mathbb{R}^n$. Since it has n columns and $\mathbb{R}^n$ is n-dimensional, this is equivalent to the columns being a basis for $\mathbb{R}^n$. Thus we may simply let β denote the set of columns of A, so that β_i is the ith column of A. We claim that with this choice of β, $[I]_\beta^{\text{std}} = A$. Indeed, the ith column of $[I]_\beta^{\text{std}}$ is the coordinate vector $[I(\beta_i)]_{\text{std}} = [\beta_i]_{\text{std}}$, but β_i is a vector in $\mathbb{R}^n$, so $[\beta_i]_{\text{std}} = \beta_i$. Thus the ith column of $[I]_\beta^{\text{std}}$ is equal to β_i, which is the ith column of A by construction.

$\square$