MATH 350 Linear Algebra
Quiz 5 Solutions

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1 Solutions

(a) Consider the basis $\beta = ((2,1),(3,4))$ of $\mathbb{R}^2$. Write β_1, β_2 for the first and second basis elements respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by

$$\begin{aligned} T(\beta_1) &= 2\beta_1 \\ T(\beta_2) &= -\beta_2 \end{aligned}$$

Find $[T]_\beta$.

Solution. Recall that $[T]_\beta$ denotes the matrix for T relative to the basis β for both the domain and codomain. By definition of this matrix, its ith column is the coordinate vector of $T(\beta_i)$ with respect to β. Thus

$$[T]_\beta = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) Find $[T] = [T]_{\text{std}}$.

Recall that $[T] = [T]_{\text{std}}$ is the matrix of T with respect to the standard basis of $\mathbb{R}^2$ (for both the domain and codomain), namely $\{e_1, e_2\} = \{(1,0),(0,1)\}$. By definition, the ith column of T is just $T(e_i)$. We can calculate this in two ways.

Solution 1. We wish to write e_1, e_2 as a linear combination of β_1, β_2. Note that this is possible since $[T]_\beta$ is invertible, so T is invertible (hence onto). For e_1, this amounts to solving the equation

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or equivalently, the linear system

$$\begin{aligned} 2a + 3b &= 1 \\ a + 4b &= 0 \end{aligned}$$

We can solve this using the techniques of Math 250. The second equation forces $a = -4b$, so the first equation becomes $-8b + 3b = 1$, so $-5b = 1$, so $b = -1/5$ and $a = 4/5$. It follows that

$$T(e_1) = T(a\beta_1 + b\beta_2) = T\left(\frac{4}{5}\beta_1 - \frac{1}{5}\beta_2\right) = \frac{4}{5}T(\beta_1) - \frac{1}{5}T(\beta_2) = \frac{4}{5}(2\beta_1) - \frac{1}{5}(-\beta_2) = \frac{4}{5}\begin{bmatrix} 4 \\ 2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 19/5 \\ 12/5 \end{bmatrix}$$

For e_2, a similar calculation shows that $e_2 = \frac{-3}{5}\beta_1 + \frac{2}{5}\beta_2$. Thus,

$$T(e_2) = T\left(\frac{-3}{5}\beta_1 + \frac{2}{5}\beta_2\right) = \frac{-3}{5}T(\beta_1) + \frac{2}{5}T(\beta_2) = \frac{-3}{5}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -18/5 \\ -14/5 \end{bmatrix}$$

Thus

$$[T] = [T(e_1)\ \ T(e_2)] = \frac{1}{5}\begin{bmatrix} 19 & -18 \\ 12 & -14 \end{bmatrix}$$

Solution 2. A second solution involves considering a change of basis transformation. Consider the map $U : \mathbb{R}^2 \to \mathbb{R}^2$ given by sending $e_i \mapsto \beta_i$ (for $i = 1, 2$). Then we can compute the effect of $U^{-1}TU$ as follows:

$$\begin{array}{ccccccccc} U^{-1}TU: & \mathbb{R}^2 & \xrightarrow{U} & \mathbb{R}^2 & \xrightarrow{T} & \mathbb{R}^2 & \xrightarrow{U^{-1}} & \mathbb{R}^2 \\ & e_1 & \mapsto & \beta_1 & \mapsto & 2\beta_1 & \mapsto & 2e_1 \\ & e_2 & \mapsto & \beta_2 & \mapsto & -\beta_2 & \mapsto & -e_2 \end{array}$$

It's easy to see that the matrix of $U^{-1}TU$ with respect to the standard basis is

$$[U^{-1}TU] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

It is also easy to see that $[U] = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right]$. On the other hand, we also have

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = [U^{-1}TU] = [U^{-1}][T][U] = [U]^{-1}[T][U]$$

So left-multiplying by U and right-multiplying by U^{-1}, we get

$$[T] = [U]\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}[U]^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Performing the matrix multiplication yields

$$[T] = \begin{bmatrix} 4 & -3 \\ 2 & -4 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 19 & -18 \\ 12 & -14 \end{bmatrix}$$