MATH 350 Linear Algebra
Quiz 3 Solutions

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1. Define "linear transformation".

   Solution. A linear transformation between vector spaces V, W over F is a function $f : V \to W$ which satisfies $f(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in V$, and $f(av) = af(v)$ for all $v \in V, a \in F$.

2. Prove that if $T : V \to W$ is linear, then for any $v \in V$, $T^{-1}(T(v)) = v + \operatorname{Ker} T$.

   Recall that for any $w \in W$, $T^{-1}(w) := \{v \in V \mid T(v) = w\}$, and that $v + \operatorname{Ker} T := \{v + v' \mid v' \in \operatorname{Ker} T\} \subset V$.

   Proof. To show equality of two sets $A = B$, we must show that $A \subset B$ and $B \subset A$.

   We first show that $v + \operatorname{Ker} T \subset T^{-1}(T(v))$. Suppose $v_1 \in v + \operatorname{Ker}_T$. Then $v_1 = v + k$ for some $k \in \operatorname{Ker} T$, so $T(v_1) = T(v + k) = T(v) + T(k) = T(v) + 0 = T(v)$. This shows that $v_1 \in T^{-1}(T(v))$.

   Next we show that $T^{-1}(T(v)) \in v + \operatorname{Ker} T$. Suppose $v_1 \in T^{-1}(T(v))$. This means that $T(v_1) = T(v)$, so $T(v_1 - v) = T(v_1) - T(v) = 0$. This shows $v_1 - v \in \operatorname{Ker} T$. On the other hand, we can always write $v_1 = v + (v_1 - v) \in v + \operatorname{Ker} T$.

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