MATH 350 Linear Algebra
Quiz 2 Solutions

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September 20, 2022

1. Let V be a vector space over a field F, $S \subset V$ a subspace. Define $\operatorname{Span} S$ without using the words "linear combination".

   Solution. $\operatorname{Span} S$ is the subset of V consisting of vectors v which can be written as

   $$v = a_1 s_1 + \cdots + a_n s_n$$

   for some $a_1, \ldots, a_n \in F$ and $s_1, \ldots, s_n \in S$.

2. Let V be a vector space, and $S \subset V$ a spanning set. Let $B \subset S$ be a maximal linearly independent set[1]. Prove that B must be a basis.

   Proof. We will prove that B is a basis by contradiction. Suppose B is not a basis. Since B is linearly independent, the only way it isn't a basis is if it does not span V. This means that $\operatorname{Span} B$ cannot contain S (if $\operatorname{Span} B$ contains S, then it must contain $\operatorname{Span} S = V$). Thus there exists $s \in S$, with $s \notin \operatorname{Span} B$. This means that $B \cup \{s\}$ is linearly independent, contradicting the maximality of B. □

   Remark. The proof doesn't have to proceed by contradiction. The basic idea of the proof is the following: If B is a linearly independent set which does not span V, then we can find a vector $s \in S - B$ such that $B \cup \{s\}$ is linearly independent. In other words, any linearly independent subset of S which doesn't span V can be enlarged to a larger linearly independent subset. It follows that a maximal such subset must span V, and hence be a basis. Exercise: Write out a version of the proof without using proof by contradiction.

[1] This means that B is linearly independent, and if B' is another subset which contains B and is not equal to B, then B' is not linearly independent.