MATH 350 Linear Algebra
Quiz 1 Solutions

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1. Let V be a vector space over a field F. Let $S \subset V$ be a nonempty subset. Prove that $\text{Span}(S)$ is a vector space.

   Proof. To show that $\text{Span}(S)$ is a subspace, it suffices to check that for any $v, w \in \text{Span}(S)$ and $a \in F$, that $v + w \in \text{Span}(S)$ and $av \in \text{Span}(S)$.

   First let $v, w \in \text{Span}(S)$. By definition of $\text{Span}(S)$, there exist vectors $v_1, \ldots, v_n, w_1, \ldots, w_n \in S$, and scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ such that

   $$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n \qquad \text{and} \qquad w = b_1w_1 + b_2w_2 + \cdots + b_nw_n$$

   Since the v_i's and w_i's are in S, this shows that $v + w = a_1v_1 + \cdots + a_nv_n + b_1w_1 + \cdots + b_nw_n$ is a linear combination of vectors in S, so again by definition of $\text{Span}(S)$, this shows that $v + w \in \text{Span}(S)$.

   Similarly, $av = a(a_1v_1 + \cdots + a_nv_n) = a(a_1v_1) + \cdots + a(a_nv_n) = (aa_1)v_1 + \cdots + (aa_n)v_n$. Since $a, a_i \in F$, so is aa_i. Again since each v_i is in S, this shows that av is also a linear combination of vectors in S, so $av \in \text{Span}(S)$.

   □

   Remark 1. Here are some remarks/common mistakes.

   (a) The fact that $v + w \in \text{Span}(S)$ for any $v, w \in S$ is obvious (it's part of the definition of span). The key is to show that $v + w \in \text{Span}(S)$ for any $v, w \in \text{Span}(S)$ (and similarly with av).

   (b) The set S is not necessarily finite! Thus you can't simply write $S = \{v_1, \ldots, v_n\}$. The set S may not even be countable! (for example if $S = V = \mathbb{R}^n$), so you can't even write S as $\{v_i\}_{i \in \mathbb{N}}$.

   (c) In general the relationship between S, $\text{Span}(S)$, and V is: $S \subset \text{Span}(S) \subset V$.

2. With notations as above, show that if $W \subset V$ is a subspace containing S, then W contains $\text{Span}(S)$.

   This problem was not graded. Nonetheless, here is a solution. Please study this solution and use it as a reference for future proofs of this type.

   Proof. Let $v \in \text{Span}(S)$. Our goal is to show that $v \in W$. By definition of $\text{Span}(S)$, we may write

   $$v = a_1v_1 + \cdots + a_nv_n$$

   for some $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in S$. Our goal is to show that $v \in W$. First, since each $v_i \in S \subset W$, we have $v_i \in W$, so $a_iv_i \in W$ by the definition of subspace. It follows that v is a sum of vectors in W. Clearly if $n = 2$, then the definition of subspace would imply that $v = a_1v_1 + a_2v_2 \in W$. If $n = 3$, then you can say $v = (a_1v_1 + a_2v_2) + a_3v_3$. We already know that $a_3v_3 \in W$ and that $a_1v_1 + a_2v_2 \in W$, so their sum is also in W. It should be intuitively clear that this statement should hold for arbitrary n. To make it precise, you can use induction:

Let $P(n)$ be the statement that any linear combination of $\leq n$ vectors in S lies in W. The statement $P(1)$ is just the statement that $a_1 v_1 \in W$, which follows from the definition of subspace. This proves the base case.

For the inductive step, we must show that $P(n) \Rightarrow P(n+1)$ for any $n \geq 1$. Suppose $P(n)$ holds. To show $P(n+1)$, we must show that

$$x := a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} \in W$$

where each $a_i \in F, v_i \in S$. We may write this as

$$x = (a_1 v_1 + \cdots + a_n v_n) + a_{n+1} v_{n+1}$$

Since $P(n)$ is assumed true, $a_1 v_1 + \cdots + a_n v_n \in W$ and $a_{n+1} v_{n+1} \in W$. Thus x is a sum of two vectors in W, so $x \in W$ by the definition of subspace.

Thus we have proven $P(n)$ for all n, which implies $v \in W$, as desired. □

Remark 2. The idea of induction is that you prove two statements. First you prove $P(1)$ (the base case). Second, you prove $P(n) \Rightarrow P(n+1)$ for all $n \geq 1$ (the inductive step). Together, they imply the chain of implications $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$. Essentially, in proving these two statements, you have defined an algorithm which produces proofs of $P(n)$ for any n.[1] Induction is a fundamental concept in mathematics, and an indispensible tool when writing proofs, or more generally in understanding why things are true. You should study this proof until it is second nature.

[1] Whether this constitutes a proof of $\forall n, P(n)$ is an interesting philosophical question. In this class we will assume that it does.