Key

MATH 350:02 FINAL EXAM

May 11, 2022

NAME (*please print*): ____________________

SIGNATURE: ____________________

Do all 8 problems.

Note that some of the problems have several parts.

Show all your work and justify your answers.

Good luck!

Problem number	Possible points	Points earned (out of 100):
1	10	
2	5	
3	5	
4	20	
5	15	
6	10	
7	15	
8	20	
Total points earned:		

See pf. of Thm. 2 on p. 28 of Lecture #13 notes, 3/2/22

(10) 1. Suppose that the vectors $v_1, v_2, \ldots, v_k$ span $\mathbb{F}^n$. ($\mathbb{F}$ is a field.)

5 (a) Prove that $k \geq n$.

5 (b) Prove that if $k = n$, then $\{v_1, v_2, \ldots, v_k\}$ is a basis of $\mathbb{F}^n$.

(5) 2. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Suppose that the null space of T is $\{0\}$ and suppose that $\{u, v, w\}$ is a linearly independent subset of V. Show that $\{T(u), T(v), T(w)\}$ is a linearly independent subset of W.

Suppose

$$aT(u) + bT(v) + cT(w) = 0$$

Then

$$T(au + bv + cw) = 0,$$

so

$$au + bv + cw = 0,$$

so

$$a = b = c = 0.$$

(5) 3. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Assume that T is invertible, that is, that T is one-to-one and onto. Prove that $T^{-1} : W \to V$ is linear.

See pf. of Thm. 2.17, p. 101.

(20) 4. **NOTE: This problem has six parts. Parts (a)-(c) are on this page and parts (d)-(f) are on the following pages.**

Let $\beta = \{1, x, x^2\}$ be the standard ordered basis of $P_2(\mathbb{R})$.

Let $\beta' = \{f_1(x), f_2(x), f_3(x)\}$, where $f_1(x) = x + 2x^2$, $f_2(x) = 1 + x^2$ and $f_3(x) = 1$.

Note that β and β' will be used throughout this problem.

3 (a) Show that β' is a basis of $P_2(\mathbb{R})$.

(Can use (b).)

2 (b) Find the change-of-coordinate matrix Q that changes β'-coordinates into β-coordinates.

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

$(\det Q = 1)$

3 (c) Find Q^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right) \text{ so } Q^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

CONTINUATION OF PROBLEM 4

4

(d) Let $g(x) = 5 + x - 3x^2$.

Find the column vector $[g(x)]_\beta$, the coordinate vector of $g(x)$ with respect to β.

Also find the coordinate vector $[g(x)]_{\beta'}$ of $g(x)$ with respect to β'.

2 $[g(x)]_\beta = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$

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2 $[g(x)]_{\beta'}$:

$$5 + x - 3x^2 = a(x+2x^2) + b(1+x^2) + c(1)$$
$$= (b+c) + ax + (2a+b)x^2$$

$5 = b + c$
$1 = a$
$-3 = 2a + b$

$a = 1$
$b = -5$
$c = 10$

$$[g(x)]_{\beta'} = \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix}$$

Or: $[g(x)]_{\beta'} = Q^{-1}[g(x)]_\beta$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix}$$

CONTINUATION OF PROBLEM 4

4

(e) Let T be the linear operator on $P_2(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = (2a + c) + bx + (a + b)x^2.$$

Find the matrix $[T]_\beta$ of T with respect to β.

Also find the matrix $[T]_{\beta'}$ of T with respect to β'.

2 $$[T]_\beta = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

2 $$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

Or could compute $[T]_{\beta'}$ directly.

CONTINUATION OF PROBLEM 4

4 (f) Find the coordinate vector $[T(g(x))]_\beta$ of $T(g(x))$ with respect to β.

Also find the coordinate vector $[T(g(x))]_{\beta'}$ of $T(g(x))$ with respect to β'.

(Recall that $g(x)$ was defined in Part (d).)

2 $$[T(g(x))]_\beta = [T]_\beta [g(x)]_\beta$$

$$= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix}$$

Or, $T(g(x)) = 7 + x + 6x^2$ directly

2 $$[T(g(x))]_{\beta'} = [T]_{\beta'} [g(x)]_{\beta'}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

Or, could expand $T(g(x))$ in terms of β' directly.

Or, $$[T(g(x))]_{\beta'} = Q^{-1} [T(g(x))]_\beta$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

(15) 5. **NOTE: This problem has five parts. Parts (c), (d) and (e) are on the following pages.**

Consider the real 3×3 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

3 (a) Find the characteristic polynomial of A.

$$\det(A - tI_3) = \det\begin{pmatrix} 2-t & 0 & 0 \\ -1 & 3-t & 1 \\ 0 & -1 & 1-t \end{pmatrix}$$

$$= (2-t)((3-t)(1-t)+1)$$

$$= (2-t)(t^2-4t+4)$$

$$= (2-t)(t-2)^2$$

$$= -(t-2)^3$$

2 (b) Find all of the eigenvalues of A.

$\lambda = 2$

CONTINUATION OF PROBLEM 5

3

(c) Find all of the eigenvectors of A corresponding to each eigenvalue.

$$A - 2I_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

All of the <u>nonzero</u> multiples of

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

-1 if don't say nonzero

2

(d) Write down the Jordan canonical form J of A.

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

CONTINUATION OF PROBLEM 5

5 (e) Find an invertible matrix Q such that $Q^{-1}AQ = J$. (You do *not* have to calculate Q^{-1}.)

$$(A - 2I_3)u = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$x_3 = 0$ gives $u = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$(A - 2I_3)v = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$x_3 = 0$ gives $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Can take $Q = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(10) 6. Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}$ and let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Show that $\det(A + tI_n) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

(Hints: Use mathematical induction on n. Use cofactor expansion along the first row.)

See HW #7 solutions,
4.3 #24

(15) 7. Consider the inner product space $V = P_1(\mathbb{R})$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$.

5 (a) Use the Gram-Schmidt process to find an orthogonal basis of V starting from the standard ordered basis $\{1, x\}$.

$$v_1 = 1$$

$$v_2 = x - \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1$$

$$\langle x, 1\rangle = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\langle 1, 1\rangle = \int_0^1 1\,dt = [t]_0^1 = 1$$

$$v_2 = x - \frac{1/2}{1} 1 = x - \frac{1}{2}$$

5 (b) Normalize your basis elements to obtain an orthonormal basis β for V.

$$\langle v_1, v_1\rangle = \langle 1, 1\rangle = 1$$

$$\langle v_2, v_2\rangle = \langle x - \tfrac{1}{2}, x - \tfrac{1}{2}\rangle$$

$$= \int_0^1 (t - \tfrac{1}{2})^2 dt = \int_0^1 (t^2 - t + \tfrac{1}{4}) dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{4}{12} - \frac{6}{12} + \frac{3}{12} = \frac{1}{12}$$

$$\frac{v_1}{\|v_1\|} = v_1 = 1 \qquad \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{\frac{1}{12}}} = \sqrt{12}\, v_2 = 2\sqrt{3}\, v_2$$

$$= \sqrt{3}(2x - 1)$$

NOTE: Part (c) is on the next page.

$$\beta = \{1, \sqrt{3}(2x-1)\}$$

CONTINUATION OF PROBLEM 7

5

(c) Find the Fourier coefficients of $h(x) = 1 - 2x$ with respect to your orthonormal basis β.

$$\langle h(x), v_1 \rangle = \langle 1-2x, 1 \rangle$$

$$= \int_0^1 (1-2t)(1)\,dt = \int_0^1 (1-2t)\,dt$$

$$= \left[t - t^2\right]_0^1 = 0$$

$$\langle h(x), v_2 \rangle = \langle 1-2x, \sqrt{3}(2x-1) \rangle$$

$$= \int_0^1 (1-2t)\sqrt{3}(2t-1)\,dt$$

$$= -\sqrt{3}\int_0^1 (2t-1)^2\,dt = -\sqrt{3}\int_0^1 (4t^2-4t+1)\,dt$$

$$= -\sqrt{3}\left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1$$

$$= -\sqrt{3}\left(\frac{4}{3} - 2 + 1\right) = -\sqrt{3}\left(\frac{1}{3}\right)$$

$$= -\frac{\sqrt{3}}{3} \quad \left(\text{or } -\frac{1}{\sqrt{3}}\right)$$

(20) 8. Consider the complex 3×3 matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}.$$

2 (a) Is A self-adjoint? Justify your answer.

$$A^* = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} \neq A$$

No.

2 (b) Is A normal? Justify your answer.

$$A^*A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$AA^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= A^*A$$

Yes.

NOTE: Part (c) is on the next page.

CONTINUATION OF PROBLEM 8

(c) Find a unitary matrix P and a diagonal matrix D such that $P^* AP = D$.

(3) $\det(A - tI_3) = \det\begin{pmatrix} -t & -1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 2i-t \end{pmatrix}$

$= (2i - t)(t^2 + 1) = -(t - 2i)(t - i)(t + i)$

(3) $\lambda_1 = 2i$: $\begin{pmatrix} -2i & -1 & 0 \\ 1 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Take $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (or any norm-1 multiple)

(3) $\lambda_2 = i$: $\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Take $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ (or any norm-1 multiple)

(3) $\lambda_3 = -i$: $\begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 3i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Take $v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$ (or any norm-1 multiple)

With these choices

(2) $P = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}$ (2) $D = \begin{pmatrix} 2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$

END OF EXAM