(10) 1. Suppose that the vectors $v_1, v_2, \ldots, v_k$ span $\mathbb{F}^n$. ($\mathbb{F}$ is a field.)

(a) Prove that $k \geq n$.

(b) Prove that if $k = n$, then $\{v_1, v_2, \ldots, v_k\}$ is a basis of $\mathbb{F}^n$.

(5) 2. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Suppose that the null space of T is $\{0\}$ and suppose that $\{u, v, w\}$ is a linearly independent subset of V. Show that $\{T(u), T(v), T(w)\}$ is a linearly independent subset of W.

(5) 3. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Assume that T is invertible, that is, that T is one-to-one and onto. Prove that $T^{-1} : W \to V$ is linear.

(20) 4. **NOTE: This problem has six parts. Parts (a)-(c) are on this page and parts (d)-(f) are on the following pages.**

Let $\beta = \{1, x, x^2\}$ be the standard ordered basis of $P_2(\mathbb{R})$.

Let $\beta' = \{f_1(x), f_2(x), f_3(x)\}$, where $f_1(x) = x + 2x^2$, $f_2(x) = 1 + x^2$ and $f_3(x) = 1$.

Note that β and β' will be used throughout this problem.

(a) Show that β' is a basis of $P_2(\mathbb{R})$.

(b) Find the change-of-coordinate matrix Q that changes β'-coordinates into β-coordinates.

(c) Find Q^{-1}.

CONTINUATION OF PROBLEM 4

(d) Let $g(x) = 5 + x - 3x^2$.

Find the column vector $[g(x)]_\beta$, the coordinate vector of $g(x)$ with respect to β.

Also find the coordinate vector $[g(x)]_{\beta'}$ of $g(x)$ with respect to β'.

CONTINUATION OF PROBLEM 4

(e) Let T be the linear operator on $P_2(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = (2a + c) + bx + (a + b)x^2.$$

Find the matrix $[T]_\beta$ of T with respect to β.

Also find the matrix $[T]_{\beta'}$ of T with respect to β'.

CONTINUATION OF PROBLEM 4

(f) Find the coordinate vector $[T(g(x))]_\beta$ of $T(g(x))$ with respect to β.

Also find the coordinate vector $[T(g(x))]_{\beta'}$ of $T(g(x))$ with respect to β'.

(Recall that $g(x)$ was defined in Part (d).)

(15) 5. **NOTE: This problem has five parts. Parts (c), (d) and (e) are on the following pages.**

Consider the real 3×3 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Find all of the eigenvalues of A.

CONTINUATION OF PROBLEM 5

(c) Find all of the eigenvectors of A corresponding to each eigenvalue.

(d) Write down the Jordan canonical form J of A.

CONTINUATION OF PROBLEM 5

(e) Find an invertible matrix Q such that $Q^{-1}AQ = J$. (You do *not* have to calculate Q^{-1}.)

(10) 6. Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}$ and let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Show that $\det(A + tI_n) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

(Hints: Use mathematical induction on n. Use cofactor expansion along the first row.)

(15) 7. Consider the inner product space $V = P_1(\mathbb{R})$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$.

(a) Use the Gram-Schmidt process to find an orthogonal basis of V starting from the standard ordered basis $\{1, x\}$.

(b) Normalize your basis elements to obtain an orthonormal basis β for V.

NOTE: Part (c) is on the next page.

CONTINUATION OF PROBLEM 7

(c) Find the Fourier coefficients of $h(x) = 1 - 2x$ with respect to your orthonormal basis β.

(20) 8. Consider the complex 3×3 matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}.$$

(a) Is A self-adjoint? Justify your answer.

(b) Is A normal? Justify your answer.

NOTE: Part (c) is on the next page.

CONTINUATION OF PROBLEM 8

(c) Find a unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

END OF EXAM

MATH 350:02 FINAL EXAM

May 11, 2022

NAME (*please print*): ______________________________

SIGNATURE: ______________________________

Do all 8 problems.

Note that some of the problems have several parts.

Show all your work and justify your answers.

Good luck!

Problem number	Possible points	Points earned (out of 100):
1	10	
2	5	
3	5	
4	20	
5	15	
6	10	
7	15	
8	20	
Total points earned:		