Solutions and comments

MATH 350:02, EXAM 2

April 13, 2022

NAME (*please print*): ______________________________

SIGNATURE: ______________________________

Do all 5 problems.

Show all your work and justify your answers.

Problem number	Possible points	Points earned (out of 100):
1	20	
2	20	
3	20	
4	20	
5	20	
Total points earned:		

(20) 1. Let $v_1, v_2, \ldots, v_k$ be linearly independent vectors in $\mathbb{F}^n$.

(a) Prove that $k \leq n$.

As in the Lecture #13 notes, pages 26–27:
Form $A = (\underline{v}_1 \cdots \underline{v}_k) \in M_{n \times k}(\mathbb{F})$.
Let $R = \mathrm{rref}(A)$. By the Column Correspondence Principle, the k cols. of R are linearly indep. Thus $R\underline{x} = \underline{0}$ has no nonzero solutions, so nullity$(R) = 0$. Thus R must be of the form $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$ (or just I_k).

Since R has n rows, $k \leq n$.

Interesting note: The logic here does not require knowing that $\mathrm{rref}(A)$ is unique, only that an RREF for A exists.

(b) Prove that if $k = n$, then $\{v_1, v_2, \ldots, v_k\}$ is a basis of $\mathbb{F}^n$.

From above, R must be $I_k = I_n$.

Thus rank$(R) = n$, so rank$(A) = $ rank$(R) = n$. Thus Col$(A) = \mathbb{F}^n$, that is, the span of the columns of A is $\mathbb{F}^n$.

So $\{\underline{v}_1, \ldots, \underline{v}_n\}$ spans $\mathbb{F}^n$ and thus is a basis.

To elaborate: Since R has no zero rows, $R\underline{x} = \underline{c}$ is consistent $\forall \underline{c} \in \mathbb{F}^n$. Thus $A\underline{x} = \underline{b}$ is consistent $\forall \underline{b} \in \mathbb{F}^n$. Thus the columns of A span $\mathbb{F}^n$. (The Column Corresp. Principle is not relevant here.)

(20) 2. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and take $\mathbb{F} = \mathbb{R}$.

(a) Find the eigenvalues of A and find an ordered basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

$$\det(A - tI_2) = \det\begin{pmatrix} 1-t & 2 \\ 3 & 2-t \end{pmatrix} = (1-t)(2-t) - 6$$
$$= (t-4)(t+1), \text{ so } \lambda_1 = 4, \ \lambda_2 = -1$$

For $\lambda_1 = 4$:

$$\begin{pmatrix} 1-4 & 2 \\ 3 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1$:

$$\begin{pmatrix} 1+1 & 2 \\ 3 & 2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\left\{\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ (or any rescalings)

(Or: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}\right\}$ if λ_1 and λ_2 are reversed.)

(b) Write down an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. (You do *not* have to calculate Q^{-1}.)

$$Q = \begin{pmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

(depending on the choices in (a))

(20) 3. Let T be a linear operator on a finite-dimensional vector space V over $\mathbb{F}$.

(a) Prove that T is invertible if and only if 0 is not an eigenvalue of T.

This is 5.1 #9(a). See HW #8 solutions.

(b) Prove that if T is invertible, then a scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

This is 5.1 #9(b). Again, see HW #8 solutions.

Similar to 5.4 #6(b) and #9(b)
See HW #9 solutions.

(20) 4. Let $V = P_2(\mathbb{R})$, define the linear operator $T : V \to V$ by $T(f(x)) = f'(x) - f(1)$ for $f(x) \in V$, and let W be the T-cyclic subspace of V generated by the polynomial $x^2 - 1 \in V$.

(a) Find an ordered basis for W.

$T(x^2-1) = 2x - 0 = 2x$

$T^2(x^2-1) = T(2x) = 2 - 2 = 0$

Since $\{x^2-1, T(x^2-1)\}$ is a linearly independent set while $T^2(x^2-1)$ is a linear combination of these vectors, and in fact,

$(0)(x^2-1) + (0)(T(x^2-1)) + T^2(x^2-1) = 0,$

$\{x^2-1, 2x\}$ is an ordered basis for W.

(b) Find the characteristic polynomial of the restriction T_W of T to W.

Use either of the two methods in 5.4 #9(b).

<u>Method 1</u>: From Thm. 5.21, the char. poly. of T_W is $(-1)^2(0 + 0t + t^2) = t^2$

<u>Method 2</u>: Use the ordered basis $\beta = \{x^2-1, 2x\}$ for W. Since $T(x^2-1) = 2x$ and $T(2x) = 0$, we have: $[T]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus the char. poly. of T_W is the char. poly of $[T]_\beta$, which is t^2.

(20) 5. NOTE: PARTS (B) AND (C) OF THIS PROBLEM ARE ON THE NEXT PAGE.

Let $A = \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix}$ in $M_{2\times 2}(\mathbb{R})$.

(a) Find an eigenvector for A.

$$\det\begin{pmatrix} -t & -4 \\ 1 & 4-t \end{pmatrix} = (t-2)^2$$

$\lambda = 2$

$$\begin{pmatrix} -2 & -4 \\ 1 & 4-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (or any rescaling) is an eigvec. for $\lambda = 2$).

CONTINUATION FROM THE PREVIOUS PAGE

(b) *Use your answer to (a)* to find a Jordan canonical basis for the linear operator L_A. In other words, find an ordered basis of $\mathbb{R}^2$ which is a cycle of generalized eigenvectors of L_A.

Let $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (or whatever rescaling was chosen in (a)). We need to find $v_2 \in \mathbb{R}^2$ such that $(A - 2I_2)v_2 = v_1$

$$\begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & -4 & -2 \\ 1 & 2 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Let's take $x_2 = 0$ to get the particular soln. $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of the inhomogeneous equation. With these choices, our cycle of generalized eigvecs of L_A (with $\lambda = 2$) is $\{v_1, v_2\} = \left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, our (ordered) Jordan canonical basis.

(c) Write down the Jordan canonical form J of A and an invertible matrix Q such that $Q^{-1}AQ = J$. (You do *not* have to calculate Q^{-1}.)

$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (which is unique), and with our choices, $Q = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.