MATH 350:02, EXAM 1

February 23, 2022

NAME (*please print*): ______________________________

SIGNATURE: ______________________________

Do all 7 problems.

Show all your work and justify your answers.

Problem number	Possible points	Points earned (out of 100):
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total points earned:		

Solutions and comments

I've included some extra discussion and some alternative solutions. Be sure to understand all of the details. Of course, you didn't have to quote any theorems by theorem number. Note that I asked you to prove three theorems from your list of theorems whose proofs you need to know.

(15) 8 1. (a) Let W be the subset of $\mathbb{R}^2$ consisting of all the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $a = 3b$, that is, $W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; a = 3b \right\}$. Is W a subspace of $\mathbb{R}^2$? Justify your answer.

Yes. Verify (a), (b), (c) in Thm. 1.3:

(a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W$ because $0 = 3(0)$

W is closed under addn.

(b) Let $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \in W$. $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$

$a + a' = 3b + 3b' = 3(b+b')$ so $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} \in W$

W is closed under scalar mult.

(c) Let $\begin{pmatrix} a \\ b \end{pmatrix} \in W$, $c \in \mathbb{R}$. $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$

$ca = c(3b) = 3(cb)$ so $c\begin{pmatrix} a \\ b \end{pmatrix} \in W$

<u>Or</u>: $W = \left\{ b\begin{pmatrix} 3 \\ 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$ and so $W = \text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$. Thus W is a subspace of $\mathbb{R}^2$ by Thm. 1.5.

7 (b) Let W be the subset of $P_2(\mathbb{R})$ consisting of the polynomials $f(x)$ of the form $a_0 + a_2x^2$ where $a_0, a_2 \in \mathbb{R}$ and $a_2 = a_0 + 1$. Is W a subspace of $P_2(\mathbb{R})$? Justify your answer.

No.

$0 + 0x^2 \notin W$, so the zero vector of $P_2(\mathbb{R})$ is not in W, violating condition (a) in Thm. 1.3.

<u>Or</u>, you could give a counterexample to condition (b), such as: $x^2 \in W$ but $x^2 + x^2 \notin W$.

<u>Or</u>, you could construct a counterexl. to condn. (c), such as: $x^2 \in W$ but $2x^2 \notin W$.

(15) 2. Find a basis for the subspace W of $P_4(\mathbb{R})$ spanned by $\{x^2+1, 2x, (x+1)^2\}$. You can use any (valid) method but you must fully justify your answer.

Note that $(x+1)^2 = (x^2+1)+2x$. Thus $(x+1)^2$ is in the span of x^2+1 and $2x$. But $\{x^2+1, 2x\}$ is linearly independent because if $a(x^2+1)+b(2x)=0$, where $a, b \in \mathbb{R}$, we have $ax^2+2bx+a=0$, so $a=b=0$. Thus $\{x^2+1, 2x\}$ is a basis for W.

Or, you could use the standard basis $\{1, x, x^2, x^3, x^4\}$ of $P_4(\mathbb{R})$ (or the ordered basis $\{x^4, x^3, x^2, x, 1\}$) to identify elements of $P_4(\mathbb{R})$ with column vectors. The problem becomes: Find a basis for

$$\text{span}\left\{\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}\right\},$$

in other words, find a basis for $Col(A)$, where

$$A = \begin{pmatrix}1&0&1\\0&2&2\\1&0&1\\0&0&0\\0&0&0\end{pmatrix}. \text{ But } A \xrightarrow{RREF} \begin{pmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

so the first two columns of A are the pivot columns. Thus the first two of the three column vectors form a basis. Thus the two polynomials $1+x^2$, $2x$ form a basis for W.

(15) 3. Let V and W be vector spaces over a field $\mathbb{F}$ and let $T: V \to W$ be a linear transformation.

6 (a) Prove that $T(0_V) = 0_W$, where 0_V and 0_W are the zero vectors in V and W, respectively. (Be sure to justify your steps.)

(This is property 1 of a linear transf., p. 65, on your list. We proved it in detail in class, with discussion, in two different ways.)

Proof #1 Choose any $v \in V$. Then $T(0_V) = T(0v)$ (by Thm. 1.2(a)) $= 0\,T(v)$ (since T is linear) $= 0_W$ (again by Thm. 1.2(a)).

Pf. #2 $T(0_V) = T(0_V + 0_V)$ (by (VS3)) $= T(0_V) + T(0_V)$ (since T is linear). So $T(0_V) + 0_W = T(0_V) + T(0_V)$ (by (VS3)). Thus $T(0_V) = 0_W$ by the (left) cancellation theorem.

2 (b) Define the *null space* $N(T)$ of T.

$N(T) = \{v \in V \mid T(v) = 0_W\}$

7 (c) Prove that T is one-to-one if and only if $N(T) = \{0_V\}$.

(This thm. is on your list. We proved it in detail in class.)

Suppose that T is one-to-one. Let $v \in V$ and suppose that $T(v) = 0_W$. Since also $T(0_V) = 0_W$ (by (a) above), $v = 0_V$. Thus $N(T) = \{0_V\}$.

Conversely, suppose that $N(T) = \{0_V\}$. Let $x, y \in V$ such that $T(x) = T(y)$. Then $T(x-y) = T(x) - T(y)$ (by property 3 of a linear transf., p. 65) $= 0_W$. Thus $x - y = 0_V$ since $N(T) = \{0_V\}$ and so $x = y$. Thus T is one-to-one.

(15) 4. Let V be a vector space over a field $\mathbb{F}$ and let $u_1, u_2, \ldots, u_n$ be distinct vectors in V Assume that each vector in V can be uniquely expressed as a linear combination of vectors in the set $\beta = \{u_1, u_2, \ldots, u_n\}$. Prove that β is a basis for V.

(This is part of Thm. 1.8, on your list. We proved it in detail in class.)

Since every vector in V can be expressed in at least one way as a linear comb. of vectors in β, β spans V.

β is linearly independent:

Suppose that

$$a_1 u_1 + \cdots + a_n u_n = 0_V$$

for some $a_1, \ldots, a_n \in \mathbb{F}$.

Since, also (by Thm. 1.2(a)),

$$0 u_1 + \cdots + 0 u_n = 0_V,$$

we must have

$$a_1 = 0, \ \ldots, a_n = 0$$

by the uniqueness.

Thus β is a basis for V.

(15) 5. Define the linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^3$ by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(0)+f(1) \end{pmatrix}.$$

(T is linear but you don't have to prove this.)

8 (a) Find a basis for the range $R(T)$ of T.

$T(1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $T(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, (We're using the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$)

$T(x^2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $T(x^3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Thus $R(T)$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. But these two vectors are linearly independent, so they form a basis for $R(T)$.

(To see that these two vectors form a basis for $R(T)$, you could alternatively use the second approach for Problem 2 above):

$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ so the first 2 columns are the pivot columns, so the first two vectors form a basis for $\underset{= Col(A)}{R(T)}$

2 (b) Use your result to determine the rank of T.

$\text{rank}(T) = \dim(R(T)) = 2.$

4 (c) Use the Dimension Theorem to determine the nullity of T.

$$\text{nullity}(T) + \text{rank}(T) = \dim(P_3(\mathbb{R})) = 4$$

$$\text{So nullity}(T) = 2.$$

Note that $A = [T]_\beta^\gamma$, where $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(15) 6. Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ a_1 + 2a_2 \end{pmatrix}.$$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard ordered basis of $\mathbb{R}^2$, and also consider the ordered basis $\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.

3 (a) Write down a matrix A such that $T = L_A$. (L_A means "left multiplication by A.")

$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ (since $A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ a_1 + 2a_2 \end{pmatrix}$)

6 (b) Find the matrix $[T]_\beta$ of T with respect to the standard ordered basis β. (Note that $[T]_\beta$ can also be written as $[T]_\beta^\beta$.) You must justify your answer.

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the first column of $[T]_\beta$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, so the second column of $[T]_\beta$ is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Thus $[T]_\beta = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ (which of course agrees with A)

6 (c) Find the matrix $[T]_\beta^{\beta'}$ of T with respect to the ordered bases β and β'. As usual, justify your answer.

Label the two vectors in β' as:

$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v$, so the first column of $[T]_\beta^{\beta'}$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=u} + b\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{=v}$. Find a and b:

$-1 = a + b$

$2 = b$

So $a = -3$, $b = 2$, so $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -3u + 2v$.

Thus the second column of $[T]_\beta^{\beta'}$ is $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

We get: $[T]_\beta^{\beta'} = \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$

(10) 7. Let $\mathbb{F}$ be a field.

(a) Are the vector spaces $\mathbb{F}^6$ and $M_{2\times 3}(\mathbb{F})$ isomorphic? Justify your answer briefly.

5

Yes, because they both have dimension 6.

(See Thm. 2.19.)

(b) Are the vector spaces $P_5(\mathbb{F})$ and $\mathbb{F}^5$ isomorphic? Justify your answer briefly.

5

No, because their dimensions are different, namely, 6 and 5, respectively.

(See Thm. 2.19.)