(15) 1. (a) Let W be the subset of $\mathbb{R}^2$ consisting of all the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $a = 3b$, that is, $W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; a = 3b \right\}$. Is W a subspace of $\mathbb{R}^2$? Justify your answer.

(b) Let W be the subset of $P_2(\mathbb{R})$ consisting of the polynomials $f(x)$ of the form $a_0 + a_2x^2$ where $a_0, a_2 \in \mathbb{R}$ and $a_2 = a_0 + 1$. Is W a subspace of $P_2(\mathbb{R})$? Justify your answer.

(15) 2. Find a basis for the subspace W of $P_4(\mathbb{R})$ spanned by $\{x^2+1, 2x, (x+1)^2\}$. You can use any (valid) method but you must fully justify your answer.

(15) 3. Let V and W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation.

(a) Prove that $T(0_V) = 0_W$, where 0_V and 0_W are the zero vectors in V and W, respectively. (Be sure to justify your steps.)

(b) Define the *null space* $N(T)$ of T.

(c) Prove that T is one-to-one if and only if $N(T) = \{0_V\}$.

(15) 4. Let V be a vector space over a field $\mathbb{F}$ and let $u_1, u_2, \ldots, u_n$ be distinct vectors in V Assume that each vector in V can be uniquely expressed as a linear combination of vectors in the set $\beta = \{u_1, u_2, \ldots, u_n\}$. Prove that β is a basis for V.

(15) 5. Define the linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^3$ by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(0) + f(1) \end{pmatrix}.$$

(a) Find a basis for the range $R(T)$ of T.

(b) Use your result to determine the rank of T.

(c) Use the Dimension Theorem to determine the nullity of T.

(15) 6. Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ a_1 + 2a_2 \end{pmatrix}.$$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard ordered basis of $\mathbb{R}^2$, and also consider the ordered basis $\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.

(a) Write down a matrix A such that $T = L_A$. (L_A means "left multiplication by A.")

(b) Find the matrix $[T]_\beta$ of T with respect to the standard ordered basis β. (Note that $[T]_\beta$ can also be written as $[T]_\beta^\beta$.) You must justify your answer.

(c) Find the matrix $[T]_\beta^{\beta'}$ of T with respect to the ordered bases β and β'. As usual, justify your answer.

(10) 7. Let $\mathbb{F}$ be a field.

(a) Are the vector spaces $\mathbb{F}^6$ and $M_{2\times 3}(\mathbb{F})$ isomorphic? Justify your answer briefly.

(b) Are the vector spaces $P_5(\mathbb{F})$ and $\mathbb{F}^5$ isomorphic? Justify your answer briefly.

MATH 350:02, EXAM 1

February 23, 2022

NAME (*please print*): ____________________

SIGNATURE: ____________________

Do all 7 problems.

Show all your work and justify your answers.

Problem number	Possible points	Points earned (out of 100):
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total points earned:		