MATH 350 Linear Algebra
Homework 9

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Problems

Book Problems (2 points each, 20 points total)

- Section 7.1, Problems 2a, 2c, 3b, 5, 12
- Section 7.2, Problem 3a, 3b, 3c, 3d, 3e

Additional Problems (10 points total)

A1. Let V be a finite dimensional vector space. Suppose $V_1, \ldots, V_k$ are subspaces of V satisfying the following 2 properties

(A) For every $j \in \{1, \ldots, k\}$, $V_j \cap \sum_{i \neq j} V_i = 0$. [1]

(B) $V = \sum_{i=1}^{k} V_i$. In other words, every $v \in V$ can be written as a sum $v = v_1 + \cdots + v_k$ with each $v_i \in V_i$.

In this case, we say that V is a direct sum of $V_1, \ldots, V_k$, and we write

$$V = \bigoplus_{i=1}^{k} V_i = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

Prove the following:

(a) (2 points) The decomposition $v = v_1 + \cdots + v_k$ is unique. In other words, show that if $v = v_1' + \cdots + v_k'$ is another decomposition (with $v_i' \in V_i$), then $v_i = v_i'$ for each i. Note that the case $k = 2$ was done in homework 2 (§1.3, Problem 30).

(b) (2 points) Show that if $\beta_1, \ldots, \beta_k$ are arbitrary bases for $V_1, \ldots, V_k$ respectively, then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a basis for V.

Remark. In the original (mistaken) formulation of the problem, property (A) required $V_i \cap V_j = 0$ for every $i \neq j$. However, this is not enough to guarantee the statements in (a) and (b). Food for thought: Can you find an example of $V, V_1, \ldots, V_k$ such that the statements in (a) and (b) don't hold? Note that k will necessarily have to be ≥ 3. More food for thought: how can we see that for a linear operator $T : V \to V$, V is a direct sum of generalized eigenspaces? We know that $K_\lambda \cap K_\mu = 0$ for any $\lambda \neq \mu$, but this is not enough to ensure that it is a direct sum. It turns out property (A) is also a consequence of the statement in (a), and the statement in (a) was proven in Theorem 7.3 (in §7.1 of the book). The proof is easy in the $k = 2$ case (which we illustrated in class), but for $k \geq 3$, the proof uses some additional tools which are available in our setting.

A2. For problems A2 and A3, it may help (though not strictly necessary) to take a look at appendix E. Let $f(t) = (t-3)^2$ and $g(t) = (t-1)$. In the language of appendix E, $f(t), g(t)$ are *relatively prime polynomials.*

[1] Here, for an example, if $k = 4$ and $j = 2$, then $\sum_{i \neq 2} V_i = V_1 + V_3 + V_4 = \operatorname{Span} V_1 \cup V_3 \cup V_4$.

(a) (1 point) Find polynomials $q_1(t), r_1(t)$ with $\deg r_1(t) < \deg g(t) = 1$ such that

$$f(t) = q_1(t)g(t) + r_1(t)$$

(Hint: $r_1(t)$ should be a nonzero constant)

(b) (1 point) Using the relation $r_1(t) = f(t) - q_1(t)g(t)$, find polynomials $c(t), d(t)$ such that

$$c(t)f(t) + d(t)g(t) = 1$$

(Hint: some fractions should appear)

(c) (1 point) Let $T : V \to V$ be a linear operator on a vector space V. Verify by expanding the polynomials that

$$c(T)f(T) + d(T)g(T) = I$$

where I denotes the identity operator on V.

A3. Let $f(t) = (t-1)^3$ and $g(t) = (t-2)(t-3)$. In the language of appendix E, $f(t), g(t)$ are *relatively prime polynomials*.

(a) (1 point) Find polynomials $q_1(t), r_1(t)$ with $\deg r_1(t) < \deg g(t) = 2$ such that

$$f(t) = q_1(t)g(t) + r_1(t)$$

(Hint: $r_1(t)$ should be degree 1)

(b) (1 point) Find polynomials $q_2(t), r_2(t)$ with $\deg r_2(t) < \deg r_1(t) = 1$ such that

$$g(t) = q_2(t)r_1(t) + r_2(t)$$

(Hint: $r_2(t)$ should be nonzero and degree 0, i.e., it should be a nonzero constant. Some fractions will appear in the coefficients.)

(c) (1 point) Note that $r_1(t) = f(t) - q_1(t)g(t)$ (i.e., $r_1(t)$ is a "polynomial linear combination of $f(t), g(t)$"). Similarly, $r_2(t) = g(t) - q_2(t)r_1(t)$. Use this to find polynomials $c(t), d(t)$ such that

$$c(t)f(t) + d(t)g(t) = 1$$