MATH 350 Linear Algebra
Homework 9 Solutions

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December 13, 2022

Problems

Book Problems (2 points each, 20 points total)

- Section 7.1, Problems 2a, 2c, 3b, 5, 12
- Section 7.2, Problem 3a, 3b, 3c, 3d, 3e

Additional Problems (10 points total)

A1. Let V be a finite dimensional vector space. Suppose $V_1, \ldots, V_k$ are subspaces of V satisfying the following 2 properties

(A) For every $j \in \{1, \ldots, k\}$, $V_j \cap \sum_{i \neq j} V_i = 0$. [1]

(B) $V = \sum_{i=1}^{k} V_i$. In other words, every $v \in V$ can be written as a sum $v = v_1 + \cdots + v_k$ with each $v_i \in V_i$.

In this case, we say that V is a direct sum of $V_1, \ldots, V_k$, and we write

$$V = \bigoplus_{i=1}^{k} V_i = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

Prove the following:

(a) (2 points) The decomposition $v = v_1 + \cdots + v_k$ is unique. In other words, show that if $v = v_1' + \cdots + v_k'$ is another decomposition (with $v_i' \in V_i$), then $v_i = v_i'$ for each i. Note that the case $k = 2$ was done in homework 2 (§1.3, Problem 30).

(b) (2 points) Show that if $\beta_1, \ldots, \beta_k$ are arbitrary bases for $V_1, \ldots, V_k$ respectively, then $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a basis for V.

Remark. In the original (mistaken) formulation of the problem, property (A) required $V_i \cap V_j = 0$ for every $i \neq j$. However, this is not enough to guarantee the statements in (a) and (b). Food for thought: Can you find an example of $V, V_1, \ldots, V_k$ such that the statements in (a) and (b) don't hold? Note that k will necessarily have to be ≥ 3. More food for thought: how can we see that for a linear operator $T : V \to V$, V is a direct sum of generalized eigenspaces? We know that $K_\lambda \cap K_\mu = 0$ for any $\lambda \neq \mu$, but this is not enough to ensure that it is a direct sum. It turns out property (A) is also a consequence of the statement in (a), and the statement in (a) was proven in Theorem 7.3 (in §7.1 of the book). The proof is easy in the $k = 2$ case (which we illustrated in class), but for $k \geq 3$, the proof uses some additional tools which are available in our setting.

A2. For problems A2 and A3, it may help (though not strictly necessary) to take a look at appendix E. Let $f(t) = (t-3)^2$ and $g(t) = (t-1)$. In the language of appendix E, $f(t), g(t)$ are *relatively prime polynomials.*

[1] Here, for an example, if $k = 4$ and $j = 2$, then $\sum_{i \neq 2} V_i = V_1 + V_3 + V_4 = \operatorname{Span} V_1 \cup V_3 \cup V_4$.

(a) (1 point) Find polynomials $q_1(t), r_1(t)$ with $\deg r_1(t) < \deg g(t) = 1$ such that

$$f(t) = q_1(t)g(t) + r_1(t)$$

(Hint: $r_1(t)$ should be a nonzero constant)

(b) (1 point) Using the relation $r_1(t) = f(t) - q_1(t)g(t)$, find polynomials $c(t), d(t)$ such that

$$c(t)f(t) + d(t)g(t) = 1$$

(Hint: some fractions should appear)

(c) (1 point) Let $T : V \to V$ be a linear operator on a vector space V. Verify by expanding the polynomials that

$$c(T)f(T) + d(T)g(T) = I$$

where I denotes the identity operator on V.

A3. Let $f(t) = (t-1)^3$ and $g(t) = (t-2)(t-3)$. In the language of appendix E, $f(t), g(t)$ are *relatively prime polynomials.*

(a) (1 point) Find polynomials $q_1(t), r_1(t)$ with $\deg r_1(t) < \deg g(t) = 2$ such that

$$f(t) = q_1(t)g(t) + r_1(t)$$

(Hint: $r_1(t)$ should be degree 1)

(b) (1 point) Find polynomials $q_2(t), r_2(t)$ with $\deg r_2(t) < \deg r_1(t) = 1$ such that

$$g(t) = q_2(t)r_1(t) + r_2(t)$$

(Hint: $r_2(t)$ should be nonzero and degree 0, i.e., it should be a nonzero constant. Some fractions will appear in the coefficients.)

(c) (1 point) Note that $r_1(t) = f(t) - q_1(t)g(t)$ (i.e., $r_1(t)$ is a "polynomial linear combination of $f(t), g(t)$"). Similarly, $r_2(t) = g(t) - q_2(t)r_1(t)$. Use this to find polynomials $c(t), d(t)$ such that

$$c(t)f(t) + d(t)g(t) = 1$$

Solutions

§7.1, 2a Find the Jordan canonical form and a basis consisting of a union of disjoint cycles of generalized eigenvectors for $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

Solution. The characteristic polynomial is $\chi_A(t) = t^2 - 4t + 4 = (t-2)^2$, so the only eigenvalue is 2 with multiplicity 2. One computes that $N(A - 2I) = N\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so A has basis consisting of a single cycle of length 2 with initial vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It follows that the Jordan canonical form is

$$A^J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The end vector of our cycle can be chosen to be any v satisfying

$$(A - 2I)v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving this, we get $v = \begin{bmatrix} x \\ y \end{bmatrix}$ where $-x + y = 1$. Thus we can take $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\beta = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis consisting of cycles. One should check that $A^J = [A]_\beta$.

§7.1, 2c Find the Jordan canonical form and a basis consisting of a union of disjoint cycles of generalized eigenvectors for

$$A = \begin{bmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{bmatrix}$$

Solution. The characteristic polynomial of A is $\chi_A(t) = -t^3 + 3t^2 - 4 = (2-t)^2(1+t)$, so the eigenvalues of A are 2 with multiplicity 2, and -1 with multiplicity 1. We can compute that

$$E_2 = N(A - 2I) = N\left(\begin{bmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{bmatrix}\right)$$

is spanned by $(-1,-1,-1)$, so the dot diagram of A_{K_2} has a single column of size 2, corresponding to a single cycle of length 2. It follows that the Jordan canonical form of A is

$$A^J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We can take the initial vector of this cycle to be $b = (-1,-1,-1)$, so we can take the end vector to be any vector x satisfying $(A-2I)x = b$. We can check that

$$x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{satisfies} \qquad (A-2I)x = b$$

so we can take $\{(-1,-1,-1),(0,-1,1)\}$ to be the cycle for K_2. For $K_{-1} = E_{-1}$, a basis is $(1,3,0)$. It follows that $\beta = \{(-1,-1,-1),(0,-1,1),(1,3,0)\}$ is a basis of cycles for A. Again one should check that $[A]_\beta = A^J$.

§7.1, 3b Let V be the real vector space spanned by the set of real valued functions $\gamma = \{1, t, t^2, e^t, te^t\}$. Let $T : V \to V$ be the linear operator defined by $T(f) = f'$. Find a Jordan canonical form of T, and a basis of V consisting of a union of cycles of generalized eigenvectors.

Solution. First, one should check that the set γ is linearly independent. Note that $T(e^t) = e^t$ and $T(te^t) = e^t + te^t$. Then

$$[T]_\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is upper triangular, and so we easily see that the characteristic polynomial is $\chi_T(x) = (-x)^3(1-x)^2$, so the eigenvalues are 0 and 1 with multiplicities 3 and 2 respectively. For K_1, one easily checks that $\{e^t, te^t\} \subset K_1$ is already a cycle of generalized 1-eigenvectors. For K_0, one easily checks that $\dim N(T_{K_0}) = 1$, $\dim N(T_{K_0}^2) = 2$, and $\dim N(T_{K_0}^3) = 3$. It follows that K_0 has a dot diagram with a single column of size 3, hence admits a basis consisting of a single cycle. The end vector of this cycle can be taken to be any vector in $N(T_{K_0}^3)$ that is not in $N(T_{K_0}^2)$. For example, one can take t^2. In this case, the associated cycle is $\{2, 2t, t^2\}$. Thus,

$$\beta = \{2, 2t, t^2, e^t, te^t\}$$

is a basis of V consisting of cycles of generalized eigenvectors. The associated Jordan canonical form is

$$[T]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

§7.1, 5 Let $\gamma_1, \ldots, \gamma_p$ be cycles of generalized eigenvectors of a linear operator T corresponding to an eigenvalue λ. Prove that if the initial eigenvectors are distinct, then the cycles are disjoint.

Proof. You should prove this directly. Do not simply use Theorem 7.6. Let $\gamma_{i,1}$ denote the initial vector of the cycle γ_i. We will prove the contrapositive, that if the cycles are not disjoint, then the initial vectors cannot all be distinct. Assume the cycles are not disjoint, so that $\gamma_{i,n} = \gamma_{j,m}$ for some $i \neq j$. Then the cycles $C_{\gamma_{i,n}}, C_{\gamma_{j,m}}$ are equal. Since $\gamma_{i,1}$ is also the initial vector of the cycle $C_{\gamma_{i,n}}$, and similarly $\gamma_{j,1}$ is the initial vector of $C_{\gamma_{j,m}}$, since $C_{\gamma_{i,n}} = C_{\gamma_{j,m}}$, it follows that $\gamma_{i,1} = \gamma_{j,1}$, so the initial vectors are not distinct.

□

§7.1, 12 Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T with corresponding eigenspace E_λ and generalized eigenspace K_λ. Let U be an invertible linear operator on V that commutes with T, i.e., $TU = UT$. Prove that $U(E_\lambda) = E_\lambda$ and $U(K_\lambda) = K_\lambda$.

Proof. Suppose $v \in E_\lambda$, then $Tv = \lambda v$. Then $TUv = UTv = U\lambda v = \lambda Uv$, which is exactly to say that $Uv \in E_\lambda$.

Similarly, suppose $v \in K_\lambda$. Then for some $p \geq 1$, $(T - \lambda I)^p v = 0$. Then

$$(T - \lambda I)^p Uv = U(T - \lambda I)^p v = U0 = 0$$

which is exactly to say that $Uv \in K_\lambda$. □

Note that in the first equality in the final equation, we have used the fact that if U commutes with T, then U commutes with any polynomial in T (e.g., it commutes with $(T - \lambda I)^p$. For example, for any scalars a, b, c,

$$U(aT^2 + bT + cI) = aUT^2 + bUT + cUI = aT^2U + bTU + cIU = (aT^2 + bT + cI)U.$$

§7.2, 3a Let T be the linear operator on a finite dimensional vector space V with Jordan canonical form

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Find the characteristic polynomial of T.

Solution. Since T is similar to J, they have the same characteristic polynomials, so $\chi_T = \chi_J = (2 - t)^5(3 - t)^2$

§7.2, 3b With notation as in 3a, find the dot diagram corresponding to each eigenvalue of T.

Solution. There are two eigenvalues 2 and 3. Since J has two Jordan blocks associated to the eigenvalue 2, of sizes 3 and 2, it follows that the K_2 has a basis consisting of two cycles of generalized 2-eigenvectors, of lengths 3 and 2. Thus the dot diagram for the eigenvalue 2 takes the form

$$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{matrix}$$

For the eigenvalue 3, one should note that there are in fact two Jordan blocks, each of size 1 (not a single Jordan block of size 2!) It follows that the K_3 admits a basis consisting of two cycles of length 1. In other words, it admits an eigenbasis. Thus the dot diagram takes the form

$$\begin{matrix} \bullet & \bullet \end{matrix}$$

§7.2, 3c For which eigenvalues λ_i, if any, does $E_{\lambda_i} = K_{\lambda_i}$.

Solution. Since the dot diagram for λ_i is a basis for K_{λ_i}, and the first row is a basis for E_{λ_i}, it follows that $E_{\lambda_i} = K_{\lambda_i}$ if and only if the corresponding dot diagram consists of a single row. In our case, we find that for the eigenvalue 3, $E_3 = K_3$. This can also be computed by checking that $N(J - 3I) = N((J - 3I)^2)$.

§7.3, 3d For each eigenvalue λ_i, find the smallest positive integer p_i for which $K_{\lambda_i} = N((T - \lambda_i I)^{p_i})$.

Solution. Recall that $(T - \lambda_i I)$ moves every dot in the dot diagram up by one, and sends the top row to 0. Thus, for the eigenvalue $\lambda_1 = 2$, $p_1 = 3$, and for $\lambda_2 = 3$, $p_2 = 1$.

§7.3, 3e Compute the following numbers for each i, where U_i denotes the restriction of $T - \lambda_i I$ to K_{λ_i}.

(i) $\text{rank}(U_i)$

(ii) $\text{rank}(U_i^2)$

(iii) $\text{nullity}(U_i)$

(iv) $\text{nullity}(U_i^2)$

Solution. These can easily be read off from noting that the set of vectors associated to the dots in the dot diagram are linearly independent, and that U_i moves each dot up by one, and sends the top row to 0. For $\lambda_1 = 2$, we have $\text{rank}(U_1) = 3, \text{rank}(U_1^2) = 1, \text{nullity}(U_1) = 2, \text{nullity}(U_1^2) = 4$. One should keep in mind that rank + nullity = 5 in each case. For the eigenvalue $\lambda_2 = 3$, we have $\text{rank}(U_2) = \text{rank}(U_2^2) = 0$, and $\text{nullity}(U_2) = \text{nullity}(U_2^2) = 2$.

A1(a) Prove that the decomposition $v = v_1 + \cdots + v_k$ is unique.

Proof. Suppose $v = v_1' + \cdots + v_k'$ is another decomposition, with $v_i' \in V_i$, then we have

$$v_1 + v_2 + \cdots + v_k = v_1' + v_2' + \cdots + v_k'$$

For any $i \in \{1, \ldots, k\}$, we can rearrange this to get

$$v_i - v_i' = \sum_{j \neq i} v_j' - \sum_{j \neq i} v_j$$

where the sums on the right hand side range over all $j \in \{1, \ldots, k\}$, omitting $j = i$. The left hand side clearly lies in V_i, whereas the right hand side lies in $\sum_{j \neq i} V_j$, so property (A) implies that both the left and right hand sides are 0. I.e., $v_i = v_i'$. Since this holds for each i, this shows that the decomposition is unique.

□

A1(b) Show that if $\beta_1, \ldots, \beta_k$ are bases for $V_1, \ldots, V_k$ respectively, then the union $\beta_1 \cup \cdots \cup \beta_k$ is a basis for V.

Proof. Let $\beta := \beta_1 \cup \cdots \cup \beta_k$. If $v \in V$, then by property (B), v can be written as a sum $v_1 + \cdots + v_k$ where $v_i \in V_i$ for each i. Writing each v_i as a linear combination of elements in β_i, it follows that v is a linear combination of vectors in β. This shows that β spans V.

To see that β is linearly independent, write $\beta^1, \beta^2, \ldots, \beta^n$ be the elements of β. If

$$a_1 \beta^1 + \cdots + a_n \beta^n = 0$$

then we may group together the terms lying in each β_j. Let v_j be the sum of the terms consisting of vectors in β_j, so that $v_j \in V_j$ and $v = \sum_{j=1}^k v_j$. By A1(a), we must have that each $v_j = 0$, but since each β_j is a basis (and hence linearly independent), we find that all the coefficients of elements of β_j are 0. Since this holds for each j, it follows that all of the a_i's are 0, so β is linearly independent.

We've shown that β spans V and is linearly independent, so it is a basis for V. □

A2(a) **Solution.** $tg(t) = t(t-1) = t^2 - t$ kills the leading term of $f(t) = (t-3)^2 = t^2 - 6t + 9$. An additional $-5(t-1) = -5t + 5$ kills the second term, with remainder 4. Thus, we have

$$f(t) = (t-3)^2 = (t-5)g(t) + 4 = (t-5)(t-1) + 4$$

Thus $q_1(t) = t - 5$, $r_1(t) = 4$.

A2(b) **Solution.** Since $4 = r_1(t) = f(t) - q_1(t)g(t) = f(t) - (t-5)g(t)$, we have

$$1 = \frac{1}{4}f(t) - \frac{t-5}{4}g(t)$$

Thus we can take $c(t) = \frac{1}{4}$ and $d(t) = -\frac{t-5}{4}$.

A2(c) **Solution.** We have

$$c(T)f(T)+d(T)g(T) = \frac{1}{4}I(T-3I)^2 - \frac{1}{4}(T-5I)(T-I) = \frac{1}{4}I(T^2-6T+9I) - \frac{1}{4}(T^2-6T+5I) = \frac{9}{4}I - \frac{5}{4}I = I$$

A3(a) **Solution.** Note that $f(t) = (t-1)^3 = t^3 - 3t^2 + 3t + 1$, and $g(t) = (t-2)(t-3) = t^2 - 5t + 6$. As before, we want to match up the leading terms. We can fit $tg(t)$ into $f(t)$, with remainder $f(t) - tg(t) = 2t^2 - 3t + 1$, which means we can fit another $2g(t)$. Thus, we have

$$f(t) = (t+2)g(t) + (7t-13)$$

Thus $q_1(t) = t + 2$ and $r_1(t) = 7t - 13$.

A3(b) **Solution.** We can fit $\frac{1}{7}tr_1(t)$ into $g(t)$, with remainder

$$g(t) - \frac{1}{7}tr_1(t) = t^2 - 5t + 6 - \left(t^2 - \frac{13}{7}t\right) = -\frac{22}{7}t + 6$$

Since the degree of the remainder is not less than $r_1(t)$, we keep going. We can fit another $-\frac{22}{49}r_1(t)$ into $g(t)$, to give a joint remainder of

$$r_2(t) = g(t) - \frac{1}{7}tr_1(t) - \left(-\frac{22}{49}r_1(t)\right) = \frac{8}{49}$$

Thus, we have $q_2(t) = \frac{1}{7}t - \frac{22}{49}$ and $r_2(t) = \frac{8}{49}$.

A3(c) **Solution.** Now $r_1(t) = f(t) - q_1(t)g(t)$, and $r_2(t) = g(t) - q_2(t)r_1(t)$, so

$$\frac{8}{49} = r_2(t) = g(t) - q_2(t)(f(t) - q_1(t)g(t)) = g(t) - q_2(t)f(t) + q_2(t)q_1(t)g(t) = -q_2(t)f(t) + (1 + q_2(t)q_1(t))g(t)$$

Thus

$$1 = -\frac{49}{8}q_2(t)f(t) + \frac{49 + 49q_2(t)q_1(t))}{8}g(t)$$

In other words, we can take $c(t) = \frac{-49}{8}q_2(t)$ and $d(t) = \frac{49+49q_2(t)q_1(t)}{8}$.