MATH 350 Linear Algebra
Homework 8 Solutions

Instructor: Will Chen

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Problems

Book Problems

- Section 5.4, Problem 1 (parts a,b,c,d). One point per part, 4 points total.
- Section 5.4, 2a, 2b, 2c, 3b, 3c, 6a, 6b, 9 (for 6a, 6b), 10 (for 6a, 6b), 23. Two points each, 20 points total.

Additional Problems Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be linear. Let $\lambda_1, \ldots, \lambda_r$ be the distinct roots of the characteristic polynomial $\chi_T(t)$, with corresponding multiplicities $m_1, \ldots, m_r$ and eigenspaces $E_1, \ldots, E_r \subset V$. In other words, we can write the characteristic polynomial as

$$\chi_T(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_r - t)^{m_r} g(t) \tag{1}$$

where $g(t)$ is either 1 or a polynomial of degree ≥ 2. Since $\deg \chi_T = \dim V$ we have $0 \leq r \leq \dim V$, and $\sum_{i=1}^{r} m_i \leq \dim V$.

- In the case where $\dim V = 2$, we must have $0 \leq r \leq 2$. In the last homework, we tested various linear operators T for diagonalizability. Here is a classification of the possible situations one can encounter when in the case $\dim V = 2$.
  - Suppose $r = 2$. This means there are two eigenvalues λ_1, λ_2, so the characteristic polynomial is split. Since the muliplicities are all at least 1 and their sum is $\leq \dim V = 2$, the only possibility for the multiplicities are $m_1 = m_2 = 1$. Since $1 \leq \dim E_i \leq m_i = 1$ for $i = 1, 2$, it follows that $\dim E_i = m_i$ for $i = 1, 2$. In this case T is diagonalizable. An example of this situation is

    $$T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \chi_T(t) = (t-1)(t-2), \quad \dim E_1 = \dim E_2 = 1 = m_1 = m_2$$

  - Suppose $r = 1$. This means there is exactly one eigenvalue λ_1. Since $\chi_T(t)$ is degree 2, this means λ_1 must have multiplicity $m_1 = 2$ (or else there would be two eigenvalues). In this case we have $1 \leq \dim E_1 \leq 2$. If $\dim E_1 = 1$, then T is not diagonalizable. An example is

    $$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \chi_T(t) = (t-1)^2, \quad \dim E_1 = 1 < m_1 = 2$$

    If $\dim E_1 = 2$, then T is diagonalizable. An example is

    $$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \chi_T(t) = (t-1)^2, \quad \dim E_1 = 1 = m_1 = 1$$

  - Suppose $r = 0$. This means there are no eigenvalues, and hence T is not diagonalizable. An example of this situation is:

    $$T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \qquad \chi_T(t) = t^2 + 1$$

- (6 points) Now suppose $\dim V = 3$. Your task is to perform the analogous classification as above in the case $\dim V = 3$.
  - Suppose $r = 3$. What are the possible multiplicities m_1, m_2, m_3? For each possible triple (m_1, m_2, m_3), list the possible dimensions of the eigenspaces E_1, E_2, E_3. For each case, give an example of such a T and compute its characteristic polynomial.
  - Suppose $r = 2$. What are the possible multiplicities m_1, m_2? For each possible pair (m_1, m_2), list the possible dimensions of the eigenspaces E_1, E_2. For each case, give an example of such a T and compute its characteristic polynomial.

    For this part, it is enough to consider the eigenvalues "up to permutation". That is – you don't need to do the cases $(m_1, m_2) = (1, 2)$ and $(m_1, m_2) = (2, 1)$ separately. Just do one of them.
  - Suppose $r = 1$. What are the possibilities for m_1? For each possibility, list the possibilities for the dimension of the eigenspace E_1. For each possible dimension, give an example of such a T and compute its characteristic polynomial.
  - Suppose $r = 0$. Is this case possible? Note that we are working over $\mathbb{R}$.

  Hint. In the case $r = 3$, there is just one possibility for the multiplicities and dimensions of eigenspaces. In the case $r = 2$, there are, in total, 2 possibilities up to permutation of the eigenvalues. In the case $r = 1$, there are 4 possibilities. When looking for examples, it may help to consider upper triangular or block-diagonal matrices.

Solutions

In my solutions, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

In general, solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

§5.4, 1a True of False: There exists a linear operator T with no T-invariant subspace.

Solution. False. If $T : V \to V$, then 0 and V are always T-invariant subspaces.

§5.4, 1b True of False: If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Solution. True. This is Theorem 5.20.

§5.4, 1c True of False: Let T be a linear operator on a finite-dimensional vector space V, and let v and w be in V. If W is the T-cyclic subspace generated by v, W' is the T-cyclic subspace generated by w, and $W = W'$, then $v = w$.

Solution. False. One counterexample is the rotation by 90 degrees matrix $T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ with $V = \mathbb{R}^2$. For any nonzero vector $v \in \mathbb{R}^2$, the T-cyclic subspace generated by v is all of $\mathbb{R}^2$.

Another counterexample can be found if T has positive nullity. Let $z \in V$ be such that $z \neq 0$ but $T(z) = 0$. Then if $v \in V$ is nonzero, then one easily checks that $\langle v \rangle_T = \langle v + z \rangle_T$, but of course $v \neq v + z$.

§5.4, 1d True of False: If T is a linear operator on a finite-dimensional vector space V, then for any $v \in V$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

Solution. False. The most natural example is a polynomial space. Say, $P_3(\mathbb{R})$, and $T(f(x)) = f'(x)$. Then $\langle x^3 \rangle_T = P_3(\mathbb{R}) = \mathrm{Span}\{1, x, x^2, x^3\}$, $\langle T(x^3) \rangle_T = \langle 3x^2 \rangle_T = \mathrm{Span}\{1, x, x^2\}$.

§5.4, 2a For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

$$V = P_3(\mathbb{R}), \qquad T(f(x)) = f'(x), \qquad W = P_2(\mathbb{R})$$

Solution. Yes, W is T-invariant. Differentiation maps degree ≤ 2 polynomials to degree ≤ 2 polynomials. Note that T does not map W *onto* W, but that's okay – T-invariance doesn't require that T_W be onto.

§5.4, 2b For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

$$V = P(\mathbb{R}), \qquad T(f(x)) = xf(x), \qquad W = P_2(\mathbb{R})$$

Solution. No, W is not T-invariant. For example, $x^2 \in W$ but $T(x^2) = x^3 \notin W$.

§5.4, 2c For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

$$V = \mathbb{R}^3, \qquad T(a,b,c) = (a+b+c, a+b+c, a+b+c), \qquad W = \{(t,t,t) : t \in \mathbb{R}\}$$

Solution. Yes W is T-invariant. For $(t,t,t) \in W$, $T(t,t,t) = (3t,3t,3t) \in W$, so W is T-invariant.

§5.4, 3b Let T be a linear operator on a finite dimensional vector space V. Prove that $N(T)$ and $R(T)$ are T-invariant.

Proof. Let $v \in N(T)$. This means that $T(v) = 0$, but $0 \in N(T)$, so $T(v) \in N(T)$ for all $v \in N(T)$, so $N(T)$ is T-invariant. Next, let $v \in R(T)$. This means that $v = T(w)$ for some $w \in V$. Then $T(v) = T(T(w))$, but $T(T(w))$ is the image of $T(w)$ under T, so $T(T(w)) \in R(T)$, so $T(v) \in R(T)$, so $R(T)$ is T-invariant. □

§5.4, 3c Let T be a linear operator on a finite dimensional vector space V. Prove that E_λ is T-invariant for any eigenvalue λ of T.

Proof. Let $v \in E_\lambda$, then $T(v) = \lambda v$, which is in E_λ since E_λ is a subspace. Alternatively, we can check that λv is in E_λ by showing that $T(\lambda v) = \lambda(\lambda v)$. Indeed,

$$T(\lambda v) = \lambda T(v) = \lambda(\lambda v)$$

so $\lambda v \in E_\lambda$. □

§5.4, 6a For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

$$V = \mathbb{R}^4, \qquad T(a,b,c,d) = (a+b, b-c, a+c, a+d), \qquad z = e_1$$

Solution. We must find the minimum k such that $T^k e_1$ is a linear combination of $\{e_1, Te_1, \ldots, T^{k-1}e_1\}$. First,

$$Te_1 = (1,0,1,1),$$

which is not a linear combination of $\{e_1\}$. Next,

$$T^2 e_1 = T(Te_1) = T(1,0,1,1) = (1,-1,2,2)$$

This is not a linear combination of $\{e_1, Te_1\}$ since $T^2e_1 = (1,-1,2,2)$ has a nonzero second entry, and both e_1, Te_1 have zeros in their second entries. Next,

$$T^3 e_1 = T(T^2 e_1) = T(1,-1,2,2) = (0,-3,3,3)$$

To check if this is a linear combination of e_1, Te_1, T^2e_2, we can solve some linear equations. Here we can take a shortcut by observing that if it is, then by looking at the second entry, $(0,-3,3,3)$ must involve 3 copies of $T^2e_1 = (1,-1,2,2)$. Then we have

$$(0,-3,3,3) - 3(1,-1,2,2) = (-3,0,-3,-3) = -3Te_1$$

Thus we have

$$T^3 e_1 = 3T^2 e_1 - 3Te_1$$

Thus $\beta = \{(1,0,0,0), (1,0,1,1), (1,-1,2,2)\}$ is a basis for $\langle e_1 \rangle_T$.

§5.4, 6b Same as 6a, but using

$$V = P_3(\mathbb{R}), \qquad T(f(x)) = f''(x), \qquad z = x^3$$

Solution. As in 6a, we first note that $T(x^3) = 6x$, which is not a linear combination of $\{x^3\}$. Next, $T^2(x^3) = 0$, which is a linear combination of $\{x^3, 6x\}$, so a basis for $\langle x^3 \rangle_T$ is $\beta = \{x^3, 6x\}$.

§5.4, 9 For each operator T and invariant subspace W in 6a and 6b, compute the characteristic polynomial of T_W in two ways, as in example 6.

For 6a, the characteristic polynomial $T_{\langle e_1 \rangle_T}$ can be computed using Theorem 5.20 to be $(-1)^3(3t - 3t^2 + t^3) = -t^3 + 3t^2 - 3t = -t(t^2 - 3t + 3)$.

By using determinants, we have

$$\chi_T = \det([T]_\beta - tI_3) = \det \begin{bmatrix} -t & 0 & 0 \\ 1 & -t & -3 \\ 0 & 1 & 3-t \end{bmatrix} = (-1) \cdot 1 \det \begin{bmatrix} -t & 0 \\ 1 & -3 \end{bmatrix} + (3-t) \det \begin{bmatrix} -t & 0 \\ 1 & -t \end{bmatrix}$$

$$\cdots = -1(3t) + (3-t)(t^2) = -3t + 3t^2 - t^3$$

For 6b, the characteristic polynomial $T_{\langle x^3 \rangle_T}$ can be computed using Theorem 5.20 to be $(-1)^2(t^2) = t^2$. Using determinants, we have

$$\chi_T = \det([T]_\beta - tI_2) = \det \begin{bmatrix} -t & 0 \\ 1 & -t \end{bmatrix} = t^2$$

§5.4, 10 For each operator T in 6a and 6b, find the characteristic polynomial χ_T of T and verify that the characteristic polynomial of T_W (computed in 9) divides χ_T.

Solution. For 6a, to compute χ_T, the most universal way is just to use cofactor expansion to compute the determinant. Namely, we have

$$\chi_T(t) = \det([T]_{\text{std}} - tI_4) = \det \begin{bmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & -1 & 0 \\ 1 & 0 & 1-t & 0 \\ 1 & 0 & 0 & 1-t \end{bmatrix}$$

which isn't too bad. Cofactor expanding along the fourth column, we get

$$\chi_T(t) = (1-t) \det \begin{bmatrix} 1-t & 1 & 0 \\ 0 & 1-t & -1 \\ 1 & 0 & 1-t \end{bmatrix}$$

Cofactor expanding this determinant along the bottom row, we get

$$\chi_T(t) = (1-t) \left(1 \cdot \det \begin{bmatrix} 1 & 0 \\ 1-t & -1 \end{bmatrix} + (1-t) \det \begin{bmatrix} 1-t & 1 \\ 0 & 1-t \end{bmatrix} \right)$$

$$\cdots = (1-t)(-1 + (1-t)(1-t)^2) = (1-t)(-1 + 1 - 3t + 3t^2 - t^3) = (1-t)(-3t + 3t^2 - t^3)$$

$$\cdots = -3t + 3t^2 - t^3 + 3t^2 - 3t^3 + t^4 = -3t + 6t^2 - 4t^3 + t^4 = t(-3 + 6t - 4t^2 + t^3)$$

Using polynomial long division (see Appendix E, for example), we can find that

$$\frac{t^3 - 4t^2 + 6t - 3}{t^2 - 3t + 3} = t - 1, \qquad \text{so} \qquad (t^3 - 4t^2 + 6t - 3) = (t-1)(t^2 - 3t + 3)$$

(One can also find $t - 1$ as a quotient if you notice that 1 is a root of $t^3 - 4t^2 + 6t - 3$) It follows that

$$\chi_T = t(t^3 - 4t^2 + 6t - 3) = (t-1) \cdot \chi_{T_{\langle e_1 \rangle_T}}$$

as desired.

For 6b, to compute χ_T, again we can use cofactor expansion. Namely, using the standard basis std $= \{1, x, x^2, x^3\}$, we have

$$\chi_T(t) = \det([T]_{\text{std}} - tI_4) = \det \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 2 & 0 \\ 0 & 0 & -t & 3 \\ 0 & 0 & 0 & -t \end{bmatrix}$$

This is upper triangular, so its determinant is just the product of the diagonal entries, so we have

$$\chi_T(t) = t^4$$

which is obviously divisible by $\chi_{T_{\langle x^3 \rangle_T}} = t^2$.

§5.4, 23 Let T be a linear operator on a finite dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \dots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i. Hint: Use mathematical induction on k.

Proof. We use induction on k. The base case $k = 1$ is trivial. Now assume the statement is known for $k-1$ vectors. Let $\lambda_1, \dots, \lambda_k$ denote the distinct eigenvalues of $v_1, \dots, v_k$. Suppose $v_1 + v_2 + \cdots + v_k \in W$, then clearly

$$\lambda_k v_1 + \lambda_k v_2 + \cdots + \lambda_k v_k \in W \tag{2}$$

On the other hand, since W is T-invariant, we also have

$$T(v_1 + \cdots + v_k) = T(v_1) + \cdots + T(v_k) = \lambda_1 v_1 + \cdots + \lambda_k v_k \in W \tag{3}$$

Subtracting (2) from (3), we find that

$$(\lambda_1 - \lambda_k)v_1 + (\lambda_2 - \lambda_k)v_2 + \cdots + (\lambda_{k-1} - \lambda_k)v_{k-1} \in W$$

But each term $(\lambda_i - \lambda_k)v_i$ is just a nonzero scalar multiple of v_i, and hence is also a λ_i-eigenvector. Thus we have a sum of $k-1$ eigenvectors with distinct eigenvalues which lies in W, and hence by the induction hypothesis each must individually lie in W. Scaling as necessary, this shows that $v_1, v_2, \dots, v_{k-1} \in W$. At this point we are almost done – it remains to show that $v_k \in W$. Since $v_1, v_2, \dots, v_{k-1} \in W$, their sum $v_1 + v_2 + \cdots + v_{k-1}$ also lies in W. Since $v_1 + v_2 + \cdots + v_k \in W$, we have

$$v_k = (v_1 + v_2 + \cdots + v_k) - (v_1 + v_2 + \cdots + v_{k-1}) \in W$$

as desired. □

- (6 points) Now suppose $\dim V = 3$. Your task is to perform the analogous classification as above in the case $\dim V = 3$.
  - Suppose $r = 3$. What are the possible multiplicities m_1, m_2, m_3? For each possible triple (m_1, m_2, m_3), list the possible dimensions of the eigenspaces E_1, E_2, E_3. For each case, give an example of such a T and compute its characteristic polynomial.

    Solution. If $r = 3$, then since each $m_i \geq 1$, we must have $m_1 = m_2 = m_3 = 1$. In this case, since $1 \leq \dim E_i \leq m_i$, each eigenspace has dimension $\dim E_i = 1$. An example is:

    $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \chi_A(t) = (1-t)(2-t)(3-t)$$

  - Suppose $r = 2$. What are the possible multiplicities m_1, m_2? For each possible pair (m_1, m_2), list the possible dimensions of the eigenspaces E_1, E_2. For each case, give an example of such a T and compute its characteristic polynomial.

For this part, it is enough to consider the eigenvalues "up to permutation". That is – you don't need to do the cases $(m_1, m_2) = (1, 2)$ and $(m_1, m_2) = (2, 1)$ separately. Just do one of them.

Solution. Since the multiplicities sum to 3, the only possibility, up to permutations, is $m_1 = 1, m_2 = 2$. In this case we must have $\dim E_1 = 1$, but $\dim E_2$ could be either 1 or 2. In the first case, an example is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \chi_A(t) = (1-t)(2-t)^2$$

In the second case, an example is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

– Suppose $r = 1$. What are the possibilities for m_1? For each possibility, list the possibilities for the dimension of the eigenspace E_1. For each possible dimension, give an example of such a T and compute its characteristic polynomial.

Solution. Since the multiplicities sum to 3, we must have $m_1 = 3$. Then the dimension of E_1 could be 1, 2, or 3. For $\dim E_1 = 1$, an example is:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \chi_A(t) = (2-t)^3$$

For $\dim E_1 = 2$, an example is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \chi_A(t) = (2-t)^3$$

For $\dim E_1 = 3$, an example is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \chi_A(t) = (2-t)^3$$

– Suppose $r = 0$. Is this case possible? Note that we are working over $\mathbb{R}$.

Solution. This is not possible, since any odd degree polynomial over $\mathbb{R}$ has at least one root in $\mathbb{R}$. Phrased in another way, every linear operator on an odd-dimensional vector space over $\mathbb{R}$ has at least one eigenvalue. Note that this statement is false for even-dimensional vector spaces. For example, $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ over $\mathbb{R}$ has no eigenvalues.

Hint. In the case $r = 3$, there is just one possibility for the multiplicities and dimensions of eigenspaces. In the case $r = 2$, there are, in total, 2 possibilities up to permutation of the eigenvalues. In the case $r = 1$, there are 4 possibilities. When looking for examples, it may help to consider upper triangular or block-diagonal matrices.