MATH 350 Linear Algebra
Homework 8

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Problems

Book Problems

- Section 5.4, Problem 1 (parts a,b,c,d). One point per part, 4 points total.
- Section 5.4, 2a, 2b, 2c, 3b, 3c, 6a, 6b, 9 (for 6a, 6b), 10 (for 6a, 6b), 23. Two points each, 20 points total.

Additional Problems Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be linear. Let $\lambda_1, \ldots, \lambda_r$ be the distinct roots of the characteristic polynomial $\chi_T(t)$, with corresponding multiplicities $m_1, \ldots, m_r$ and eigenspaces $E_1, \ldots, E_r \subset V$. In other words, we can write the characteristic polynomial as

$$\chi_T(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_r - t)^{m_r} g(t) \tag{1}$$

where $g(t)$ is either 1 or a polynomial of degree ≥ 2. Since $\deg \chi_T = \dim V$ we have $0 \leq r \leq \dim V$, and $\sum_{i=1}^r m_i \leq \dim V$.

- In the case where $\dim V = 2$, we must have $0 \leq r \leq 2$. In the last homework, we tested various linear operators T for diagonalizability. Here is a classification of the possible situations one can encounter when in the case $\dim V = 2$.
  - Suppose $r = 2$. This means there are two eigenvalues λ_1, λ_2, so the characteristic polynomial is split. Since the muliplicities are all at least 1 and their sum is $\leq \dim V = 2$, the only possibility for the multiplicities are $m_1 = m_2 = 1$. Since $1 \leq \dim E_i \leq m_i = 1$ for $i = 1, 2$, it follows that $\dim E_i = m_i$ for $i = 1, 2$. In this case T is diagonalizable. An example of this situation is

    $$T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \chi_T(t) = (t-1)(t-2), \quad \dim E_1 = \dim E_2 = 1 = m_1 = m_2$$

  - Suppose $r = 1$. This means there is exactly one eigenvalue λ_1. Since $\chi_T(t)$ is degree 2, this means λ_1 must have multiplicity $m_1 = 2$ (or else there would be two eigenvalues). In this case we have $1 \leq \dim E_1 \leq 2$. If $\dim E_1 = 1$, then T is not diagonalizable. An example is

    $$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \chi_T(t) = (t-1)^2, \quad \dim E_1 = 1 < m_1 = 2$$

    If $\dim E_1 = 2$, then T is diagonalizable. An example is

    $$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \chi_T(t) = (t-1)^2, \quad \dim E_1 = 1 = m_1 = 1$$

  - Suppose $r = 0$. This means there are no eigenvalues, and hence T is not diagonalizable. An example of this situation is:

    $$T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \qquad \chi_T(t) = t^2 + 1$$

- (6 points) Now suppose $\dim V = 3$. Your task is to perform the analogous classification as above in the case $\dim V = 3$.
  - Suppose $r = 3$. What are the possible multiplicities m_1, m_2, m_3? For each possible triple (m_1, m_2, m_3), list the possible dimensions of the eigenspaces E_1, E_2, E_3. For each case, give an example of such a T and compute its characteristic polynomial.
  - Suppose $r = 2$. What are the possible multiplicities m_1, m_2? For each possible pair (m_1, m_2), list the possible dimensions of the eigenspaces E_1, E_2. For each case, give an example of such a T and compute its characteristic polynomial.

    For this part, it is enough to consider the eigenvalues "up to permutation". That is – you don't need to do the cases $(m_1, m_2) = (1, 2)$ and $(m_1, m_2) = (2, 1)$ separately. Just do one of them.
  - Suppose $r = 1$. What are the possibilities for m_1? For each possibility, list the possibilities for the dimension of the eigenspace E_1. For each possible dimension, give an example of such a T and compute its characteristic polynomial.
  - Suppose $r = 0$. Is this case possible? Note that we are working over $\mathbb{R}$.

  Hint. In the case $r = 3$, there is just one possibility for the multiplicities and dimensions of eigenspaces. In the case $r = 2$, there are, in total, 2 possibilities up to permutation of the eigenvalues. In the case $r = 1$, there are 4 possibilities. When looking for examples, it may help to consider upper triangular or block-diagonal matrices.