MATH 350 Linear Algebra
Homework 7 Solutions

Instructor: Will Chen

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Problems

Book Problems 2 points each, 26 points total

- Section 5.1, Problems 4(d), 5(b), 5(f), 9(a), 9(b), 10
- Section 5.2, Problems 2(b), 2(d), 3(d), 9(b), 11(a), 11(b), 13

  For 11(a), recall that $\mathrm{tr}(A)$ denotes the trace of the matrix A, which is defined to be the sum of the diagonal entries.

Additional Problems (2 points each, 4 points total)

- For each of the following matrices $A \in M_2(F)$, determine all eigenvalues of A. Then, for each eigenvalue λ of A, find the set of eigenvectors corresponding to λ. Then, if possible, find a basis for F^n consisting of eigenvectors of A. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

  Do the above for the "rotation by 60° matrix" $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ with $F = \mathbb{R}$. Then do the above for the same matrix but with $F = \mathbb{C}$.
- Suppose $A, B \in M_n(F)$ are similar. Recall that this means that there is an invertible matrix Q such that $B = QAQ^{-1}$. Prove that $\chi_A(t) = \chi_B(t)$.

Solutions

In my solutions, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

In general, solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

§5.1, 4d For the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix}$$

(i) Determine all eigenvalues of A, (ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ, (iii) If possible, find a basis for F^n consisting of eigenvectors of A, (iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. The characteristic polynomial is:

$$\chi_A(t) = \det \begin{bmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{bmatrix} = 2(1-t) + (-1-t)(2-t)(1-t) = (1-t)(t^2-t) = -t(t-1)^2$$

Thus, the eigenvalues of A are 0 (multiplicity 1) and 1 (multiplicty 2). For the eigenvalue 0, the eigenspace E_0 is 1-dimensional (since it has multiplicity 1), and hence it is easy to see that

$$E_0 = N(A) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$$

Thus the set of eigenvectors for the eigenvalue 0 are the nonzero vectors in E_0, or equivalently, the nonzero scalar multiples of $(1,4,2)$.

For the eigenvalue 1, the eigenspace E_1 has dimension at least 1 and at most 2 (the multiplicity). To calculate the eigenspace, we have

$$E_1 = N(A-I) = N\left(\begin{bmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{bmatrix} \right)$$

Clearly this matrix has column rank 1, and we have

$$E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus the eigenvectors for the eigenvalue 1 are the nonzero vectors in E_1, or equivalently the nonzero linear combinations of $(0,1,0)$ and $(1,0,1)$.

Remark. At this point, we can pause and notice that our calculations of E_0 and E_1 are consistent with our calculation of the characteristic polynomial. For example, if 0,1 were not roots of the characteristic polynomial, then $N(A)$ and $N(A-I)$ would have been both zero. In fact since $E_1 = N(A-I)$ is 2-dimensional, this implies that the characteristic polynomial must have the root 1 as a root of multiplicity at least 2. Since E_0 is 1-dimensional, the characteristic polynomial must the root 0 as a root of multiplicity at least 1. Since the characteristic polynomial has degree 3 and the leading term has coefficient $(-1)^3$, it follows that the characteristic polynomial must be $-t(t-1)^2$, agreeing with our calculation.

Continuing with the solution, we have found bases for E_0 and E_1. By theorem 5.5 in the book, the union of these bases is a basis for F^3. Namely, an eigenbasis is given by

$$\beta = \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Write $\beta_1, \beta_2, \beta_3$ for the vectors in β, with eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$. Let

$$Q := \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

then since each column is an eigenvector for A, we have

$$Q^{-1}AQe_i = Q^{-1}A\beta_i = Q^{-1}\lambda_i\beta_i = \lambda_i Q^{-1}\beta_i = \lambda_i e_i$$

It follows that $Q^{-1}AQ$ is diagonal with entries $0,1,1$. I.e.,

$$Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: D$$

Remark. Note, importantly (as in the solution to Quiz 7) that the invertible matrix Q is not unique. There are many other matrices Q' such that $(Q')^{-1}AQ'$ is diagonal. For example, one can permute the columns of Q, or scale the columns by scalars. One can even replace the second two columns with any other eigenbasis of E_1.

§5.1, 5b Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$$

Find the eigenvalues of T and an ordered basis β for $\mathbb{R}^3$ such that $[T]_\beta$ is a diagonal matrix.

Solution. The matrix of T (w.r.t. the standard basis) is

$$[T] = \begin{bmatrix} 7 & -4 & 10 \\ 4 & -3 & 8 \\ -2 & 1 & -2 \end{bmatrix}$$

The characteristic polynomial of T is

$$\chi_T(t) = \det(T - tI) = \det([T] - tI_3) = \det \begin{bmatrix} 7-t & -4 & 10 \\ 4 & -3-t & 8 \\ -2 & 1 & -2-t \end{bmatrix}$$

$$\cdots = -2(-32-10(-3-t))-1(8(7-t)-40)+(-2-t)((7-t)(-3-t)+16) = 64-60-20t-56+8t+40+(-2-t)(t^2-4t-5)$$

$$\cdots = -12 - 12t + (-t^3 + 2t^2 + 13t + 10) = -t^3 + 2t^2 + t - 2$$

By trial and error, one can check that $t = 1$ is a root, so $t - 1$ is a factor of $\chi_T(t)$. Using polynomial long division (or just solving the equation $(t-1)(at^2 + bt + c) = \chi_T(t)$, we find that $\frac{\chi_T(t)}{t-1} = -(t^2 - t - 2)$, so we have

$$\chi_T(t) = -(t-1)(t^2 - t - 2) = -(t-1)(t-2)(t+1)$$

It follows that the eigenvalues are 1,2,-1, each having multiplicity 1. The 1-eigenspace is

$$E_1 = N\left(\begin{bmatrix} 6 & -4 & 10 \\ 4 & -4 & 8 \\ -2 & 1 & -3 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right\}$$

Similarly, the 2-eigenspace is

$$E_2 = N\left(\begin{bmatrix} 5 & -4 & 10 \\ 4 & -5 & 8 \\ -2 & 1 & -4 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right\}$$

Finally, the (-1)-eigenspace is

$$E_{-1} = N\left(\begin{bmatrix} 8 & -4 & 10 \\ 4 & -2 & 8 \\ -2 & 1 & -1 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right\}$$

It follows that we can take β to be any basis whose elements are nonzero vectors in E_1, E_2, E_{-1}, in any order. For example, we can take

$$\beta = \left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

§5.1, 5f Let $V = P_3(\mathbb{R})$ and $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$T(f(x)) = f(x) + f(2)x$$

Find the eigenvalues of T and an ordered basis β for $P_3(\mathbb{R})$ such that $[T]_\beta$ is diagonal.

Solution. Relative to the standard basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$, the matrix of T is

$$[T]_{\text{std}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\chi_T(t) = \chi_T([T]_{\text{std}}) = \det([T]_{\text{std}} - tI_4) = \det\left(\begin{bmatrix} 1-t & 0 & 0 & 0 \\ 1 & 3-t & 4 & 8 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 1-t \end{bmatrix}\right)$$

Cofactor expanding along the bottom row repeatedly, we get

$$\chi_T(t) = (1-t)(1-t)(3-t)(1-t)$$

So the eigenvalues of T are 1 (with multiplicity 3) and 3 (with multiplicity 1). To find the eigenspaces, we can work in $\mathbb{R}^4$, viewing a general vector $a + bx + cx^2 + dx^3 \in P_3(\mathbb{R})$ as the vector $(a, b, c, d) \in \mathbb{R}^4$. The 1-eigenspace, viewed as a subspace of $\mathbb{R}^4$, is given by

$$E_1 = N\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Viewed inside $P_3(\mathbb{R})$, this implies that $\{x - 2, x^2 - 4, x^3 - 8\} \subset P_3(\mathbb{R})$ is a basis of the 1-eigenspace of T. That these vectors lie in the 1-eigenspace is easily verified using the definition of T.

The 3-eigenspace, viewed as s subspace of $\mathbb{R}^4$, is given by

$$E_3 = N\left(\begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 8 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Viewed inside $P_3(\mathbb{R})$, this implies that $\{x\}$ is a basis for the 3-eigenspace. Again this is easily checked from the definition of T. It follows that one can take β to be $\{x, x - 2, x^2 - 4, x^3 - 8\}$.

§5.1, 9a Prove that a linear operator T on a finite dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

Proof 1. Let $T : V \to V$ be linear, with $\dim V < \infty$. If 0 is an eigenvalue of T, then by definition there exists a nonzero vector v such that $T(v) = 0v = 0$. This implies that T has positive nullity, so it is not invertible. Conversely, if T is not invertible, then $N(T)$ is not the zero space, so there is a nonzero vector $v \in N(T)$. I.e., $v \neq 0$, and $T(v) = 0 = 0v$, but this says exactly that v is an eigenvector of T with eigenvalue 0, so 0 is an eigenvalue of T. □

Proof 2. Here is another proof: Zero is an eigenvalue of T if and only if 0 is a root of $\chi_T(t) = \det(T - tI)$, but that is the same as saying that $\det(T - 0I) = \det(T) = 0$, which is the same as saying that T is not invertible. □

§5.1, 9b Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof. Let $T : V \to V$ be invertible. If λ is an eigenvalue of T, then there is a nonzero $v \in V$ such that $Tv = \lambda v$, so $v = T^{-1}Tv = T^{-1}\lambda v = \lambda T^{-1}v$. Multiplying both sides by λ^{-1}, we have $\lambda^{-1}v = T^{-1}v$. Since $v \neq 0$, this says exactly that λ^{-1} is an eigenvalue of T^{-1}.

For the converse, suppose λ^{-1} is an eigenvalue of T^{-1}. This means that there is a nonzero $v \in V$, $T^{-1}v = \lambda^{-1}v$. Thus $v = TT^{-1}v = T\lambda^{-1}v = \lambda^{-1}Tv$. Multiplying both sides by λ, we get $\lambda v = Tv$. Again since $v \neq 0$, this says that λ is an eigenvalue of T. □

§5.1, 10 Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Proof. Suppose M is $n \times n$ and upper triangular. Its diagonal entries are $M_{11}, M_{22}, \dots, M_{nn}$. Then its characteristic polynomial is $\chi_M(t) = \det(M - tI)$. Since $M - tI$ is also upper-triangular, $\det(M - tI)$ is just the product of the diagonal entries of $M - tI$, which is just $M_{11} - t, M_{22} - t, \dots, M_{nn} - t$. Thus the characteristic polynomial of M is

$$\chi_M(t) = \det(M - tI) = (M_{11} - t)(M_{22} - t)\cdots(M_{nn} - t)$$

The roots of $\chi_M(t)$ are clearly just $M_{11}, M_{22}, \dots, M_{nn}$. On the other hand, the roots of $\chi_M(t)$ are also the eigenvalues of M, so it follows that the eigenvalues of M are exactly the diagonal entries of M. □

§5.2, 2b For the matrix $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right]$, test A for diagonalizability, and if it is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 3 \\ 3 & 1-t \end{bmatrix} = (1-t)^2 - 9 = t^2 - 2t - 8 = (t-4)(t+2)$$

Thus the eigenvalues are 4, -2, each of multiplicity 1. It follows that A is diagonalizable. The 4-eigenspace is

$$E_4 = N\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

and

$$E_{-2} = N\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Thus one can take

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

in which case we have

$$Q^{-1}AQ = D := \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

§5.2, 2d For the matrix

$$A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

Test A for diagonalizability, and if it is diagonalizable, find an invertible Q and diagonal D such that $Q^{-1}AQ = D$. **Solution.** The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 7-t & -4 & 0 \\ 8 & -5-t & 0 \\ 6 & -6 & 3-t \end{bmatrix}$$

Cofactor expanding along the third column, we get

$$\chi_A(t) = (3-t)((7-t)(-5-t) + 32) = (3-t)(t^2 - 2t - 3) = (3-t)(t-3)(t+1)$$

Thus the eigenvalues are 3 (multiplicity 2) and -1 (multiplicity 1). The 3-eigenspace is

$$E_3 = N\left(\begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

The (-1)-eigenspace is

$$E_{-1} = N\left(\begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\right\}$$

Since $\dim E_3$ is equal to the multiplicity, we can take

$$Q = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

In which case we have

$$Q^{-1}AQ = D := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

§5.2, 3d For the linear operator $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(f(x)) = f(0) + f(1)(x + x^2)$, test T for diagonalizability, and if it is diagonalizable, find a basis β such that $[T]_\beta$ is a diagonal matrix.

Proof. Using the standard basis $\text{std} = \{1, x, x^2\}$ of $P_2(\mathbb{R})$, the matrix of T is

$$[T]_{\text{std}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus the characteristic polynomial of T is

$$\chi_T(t) = \chi_{[T]_{\text{std}}}(t) = \det([T]_{\text{std}} - tI_3) = \det\begin{bmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{bmatrix}$$

Cofactor expanding along the top row, we find that

$$\chi_T(t) = (1-t)((1-t)^2 - 1) = (1-t)(t^2 - 2t) = (1-t)t(t-2)$$

It follows that the eigenvalues are 1, 0, 2, each of multiplicity 1. Thus T is diagonalizable. The eigenspaces are:

$$E_1 = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$$E_0 = N([T]_{\text{std}}) = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$E_2 = N\left(\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Thus we can take

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

in which case

$$Q^{-1}AQ = D := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\square$

§5.2, 9b Let T be a linear operator on a finite-dimensioanl vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper-triangular matrix. Part (a) asks you to prove that the characteristic polynomial for T splits. For this problem (part b), your task is to state *and* prove an analogous result for matrices.

Solution. The analogous statement is as follows: Let A be an $n \times n$ matrix, and suppose there is an invertible matrix Q such that $Q^{-1}AQ$ is upper-triangular. Then $\chi_A(t)$ splits.

Proof. Let $d_1, d_2, \ldots, d_n$ denote the diagonal entries of $Q^{-1}AQ$. Since $Q^{-1}AQ$ is upper-triangular, so is $Q^{-1}AQ - tI$, so $\chi_{Q^{-1}AQ}(t) = \det(Q^{-1}AQ - tI)$ is just the product of the diagonal entries of $Q^{-1}AQ - tI$, but its diagonal entries are precisely $d_1 - t, d_2 - t, \ldots, d_n - t$. Then

$$\chi_{Q^{-1}AQ}(t) = \det(Q^{-1}AQ - tI) = (d_1 - t)(d_2 - t) \cdots (d_n - t)$$

It follows that the characteristic polynomial of $Q^{-1}AQ$ is split, with roots $d_1, d_2, \ldots, d_n$. On the other hand, by the second "additional problem", we have $\chi_{Q^{-1}AQ}(t) = \chi_A(t)$. Thus since $\chi_{Q^{-1}AQ}(t)$ is split, so is $\chi_A(t)$. $\square$

§5.2, 11a Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove that $\operatorname{tr}(A) = \sum_{i=1}^k m_i \lambda_i$.

Here, recall that $\operatorname{tr}(A)$ denotes the sum of the diagonal entries of A. I.e., $\operatorname{tr}(A) = \sum_{i=1}^n A_{ii}$.

Solution 1. By hypothesis, there is an invertible matrix Q such that $Q^{-1}AQ$ is upper triangular. By §5.2 9b, this implies that $\chi_A(t)$ splits, so its roots are $\lambda_1, \ldots, \lambda_k$, with λ_i appearing m_i times. Denoting these roots as $a_1, \ldots, a_n$ (not all distinct), write $\det(A - tI) = \chi_A(t) = (a_1 - t)(a_2 - t) \cdots (a_n - t)$, so that $\sum_{i=1}^k m_i \lambda_i = \sum_{i=1}^n a_i$. Let us consider the coefficient of t^{n-1}.

On the one hand, from the expansion $\chi_A(t) = (a_1 - t) \cdots (a_n - t)$, we find that the this coefficient is exactly $(-1)^{n-1} \sum_{i=1}^n a_i$.

On the other hand, in the cofactor expansion of $\det(A - tI) = \chi_A(t)$ along the first row, we have $\det(A - tI) = \sum_{j=1}^n (-1)^{1+j} (A - tI)_{ij} \det(\widetilde{A - tI})_{1j}$. In this expansion, for each term $j > 1$, $(A - tI)_{ij}$ does not involve any t's (so it is degree 0), and the submatrix $(\widetilde{A - tI})_{1j}$ leaves out both $(A - tI)_{11}$ and $(A - tI)_{jj}$, so t appears in this submatrix at most $n - 2$ times, and so its determinant has degree at most $n - 2$. Thus, for each $j > 1$, the jth term in the cofactor expansion of $\det(A - tI)$ has at most degree $n - 2$. It follows that only the first term $(A - tI)_{11} \det(\widetilde{A - tI})_{11}$ contributes to the coefficient of t^{n-1}. By induction on n (noting that $(\widetilde{A - tI})_{11}$ is an $(n - 1) \times (n - 1)$ matrix), we find that this coefficient is precisely $(-1)^{n-1} \sum_{i=1}^n A_{ii} = (-1)^{n-1} \operatorname{tr}(A)$. It follows that we have

$$\operatorname{tr}(A) = \sum_{i=1}^n a_i = \sum_{i=1}^k m_i \lambda_i$$

as desired.

Solution 2. Alternatively, we can use the fact that the trace of similar matrices are the same. If you don't know this already, we can prove it:

Lemma 0.0.1. *For any $X, Y \in M_n(F)$, $\operatorname{tr}(XY) = \operatorname{tr}(YX)$.*

Proof. Indeed, the diagonal entries of XY, YX are:

$$(XY)_{ii} = \sum_{j=1}^{n} X_{ij}Y_{ji} \qquad (YX)_{ii} = \sum_{j=1}^{n} Y_{ij}X_{ji} = \sum_{j=1}^{n} X_{ji}Y_{ij}$$

Thus, summing over all $i = 1, 2, \ldots, n$ we get

$$\operatorname{tr}(XY) = \sum_{i=1}^{n}(XY)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}Y_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ji}Y_{ij} = \sum_{i=1}^{n}(YX)_{ii} = \operatorname{tr}(YX)$$

Here, in the third equality, we have used the observation that in both cases we are summing $X_{ab}Y_{ba}$ for all a, b between 1 and n. □

Corollary 0.0.2. *For any $Y \in M_n(F)$ and any invertible matrix $X \in M_n(F)$, we have $\operatorname{tr}(XYX^{-1}) = \operatorname{tr}(Y)$. In other words, similar matrices have the same trace.*

Proof. Group XYX^{-1} as $(XY)X^{-1}$. Then by the lemma, we have

$$\operatorname{tr}(XYX^{-1}) = \operatorname{tr}((XY)X^{-1}) = \operatorname{tr}(X^{-1}(XY)) = \operatorname{tr}(Y)$$

as desired. □

Proof of §5.2, 11a. By hypothesis, there is an invertible matrix Q such that $Q^{-1}AQ$ is upper triangular. By the second additional problem, the characteristic polynomial of A is the same as that of $Q^{-1}AQ$, so they have the same eigenvalues (with the same multiplicities). By §5.1 Problem 10, since $Q^{-1}AQ$ is upper-triangular, the eigenvalues of $Q^{-1}AQ$ are exactly the diagonal entries, with λ_i appearing m_i times. It follows that $\operatorname{tr}(Q^{-1}AQ) = \sum_{i=1}^{k} m_i\lambda_i$. By the corollary, this also shows that $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$ as desired. □

§5.2, 11b With the same assumptions as in 11a, show that $\det(A) = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}$.

By hypothesis, there is an invertible matrix Q such that $Q^{-1}AQ$ is upper triangular. By the second additional problem, the characteristic polynomial of A is the same as that of $Q^{-1}AQ$, so they have the same eigenvalues $\lambda_1, \ldots, \lambda_k$ (with the same multiplicities $m_1, \ldots, m_k$). By §5.1 Problem 10, the eigenvalues of the upper triangular matrix $Q^{-1}AQ$ are exactly the diagonal entries with each λ_i appearing m_i times. On the other hand, since $Q^{-1}AQ$ is upper-triangular, $\det(Q^{-1}AQ)$ is the product of the diagonal entries, and hence is equal to the product of the eigenvalues of $Q^{-1}AQ$ (or equivalently those of A) with multiplicity. Since determinants of similar matrices are the same, we conclude that $\det(A) = \det(Q^{-1}AQ) = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}$ as desired.

§5.2, 13 Let T be an invertible linear operator on a finite dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

Proof. The eigenspace of T corresponding to λ is $E_{T,\lambda} = N(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}$. In other words, a vector $v \in V$ lies in $E_{T,\lambda}$ if and only if $Tv = \lambda v$. Similarly, the eigenspace of T^{-1} corresponding to λ^{-1} is $E_{T^{-1},\lambda^{-1}} = N(T^{-1} - \lambda^{-1}I)$. Thus we wish to show that

$$N(T - \lambda I) = N(T^{-1} - \lambda^{-1}I)$$

Indeed, since $\lambda^{-1}T^{-1}$ is an invertible linear transformation (with inverse λT), we have

$$(T - \lambda I)v = 0 \iff (\lambda^{-1}T^{-1})(T - \lambda I)v = 0$$

But this is the same as saying...

$$\cdots \iff (\lambda^{-1}T^{-1}T - \lambda^{-1}T^{-1}\lambda I)v = 0 \iff (\lambda^{-1}I - T^{-1})v = 0 \iff (T^{-1} - \lambda^{-1}I)v = 0$$

where in the last equivalence we have used the fact that $T^{-1} - \lambda^{-1}I = -(\lambda^{-1}I - T^{-1})$. □

(b) Prove that if T is diagonalizable, then T^{-1} is also diagonalizable.

Proof. If T is diagonalizable, this means that there is a basis β for V such that $[T]_\beta$ is diagonal. But then $I_n = [I]_\beta = [TT^{-1}]_\beta = [T]_\beta [T^{-1}]_\beta$. This shows that $[T^{-1}]_\beta$ is the matrix inverse of the diagonal matrix $[T]_\beta$. Since inverses of diagonal matrices are diagonal (the diagonal entries of the inverse are just the inverses of the diagonal entries of the original matrix), this shows that $[T^{-1}]_\beta$ is diagonal, so T^{-1} is diagonalizable! $\square$

- For each of the following matrices $A \in M_2(F)$, determine all eigenvalues of A. Then, for each eigenvalue λ of A, find the set of eigenvectors corresponding to λ. Then, if possible, find a basis for F^n consisting of eigenvectors of A. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

  Do the above for the "rotation by 60° matrix" $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ with $F = \mathbb{R}$. Then do the above for the same matrix but with $F = \mathbb{C}$.

  Solution. The characteristic polynomial is

$$\chi_A(t) = \left(\frac{1}{2} - t\right)^2 + \frac{3}{4} = t^2 - t + 1$$

  By the quadratic formula, this has no roots over $\mathbb{R}$, and hence $\chi_A(t)$ is not split over $\mathbb{R}$, so A is not diagonalizable over $\mathbb{R}$. Over $\mathbb{C}$, this has the roots:

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

  each with multiplicity 1. It follows that A is diagonalizable over $\mathbb{C}$. Explicit eigenbases are:

$$E_{1/2+\sqrt{3}i/2} = N\left(\begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}i \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$$

$$E_{1/2-\sqrt{3}i/2} = N\left(\begin{bmatrix} \frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}i \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} i \\ -1 \end{bmatrix}\right\}$$

  Thus, we can take

$$Q = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}$$

  in which case we have

$$Q^{-1}AQ = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2}i & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{bmatrix}$$

- Suppose $A, B \in M_n(F)$ are similar. Recall that this means that there is an invertible matrix Q such that $B = QAQ^{-1}$. Prove that $\chi_A(t) = \chi_B(t)$.

  Proof. We have the following chain of equalities (explained below):

$$\chi_{QAQ^{-1}}(t) = \det(QAQ^{-1} - tI) = \det(QAQ^{-1} - QtIQ^{-1}) = \det(Q(A - tI)Q^{-1}) = \det(A - tI) = \chi_A(t)$$

  These equalities are not totally obvious. For the second equality we have used the fact that t represents a scalar, so $QtIQ^{-1} = tQIQ^{-1} = tQQ^{-1} = tI$. For the fourth equality, we have used the fact that determinants of similar matrices are the same. $\square$