MATH 350 Linear Algebra
Homework 6 Solutions

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Problems

Book Problems 2 points each, 20 points total

§4.2, 7 Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

by cofactor expansion along the second row.

Solution. Cofactor expansion along the second row gives:

$$\det A = (-1)^{2+1} \cdot (-1) \cdot \det \left[\begin{smallmatrix} 1 & 2 \\ 3 & 0 \end{smallmatrix}\right] + (-1)^{2+2} \cdot 0 \cdot \det \left[\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right] + (-1)^{2+3} \cdot (-3) \cdot \det \left[\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right] = -6 + 0 - 6 = -12$$

Note that because we're expanding along the second row, the terms have alternating signs, starting with a negative!

§4.2, 8 Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$$

by cofactor expansion along the third row.

Solution. Cofactor expansion along the third row gives

$$\det A = (-1)^{3+1} \cdot (-1) \cdot \det \left[\begin{smallmatrix} 0 & 2 \\ 1 & 5 \end{smallmatrix}\right] + (-1)^{3+2} \cdot 3 \cdot \det \left[\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right] + (-1)^{3+3} \cdot 0 \cdot \det \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = 2 - 15 + 0 = -13$$

§4.2, 14 Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{bmatrix}$$

Solution. Since the third column and third row has lots of zeros, a fast method is to cofactor expand along the third column or row. If we expand lalong the third row, we get:

$$\det A = (-1)^{3+1} \cdot 7 \cdot \det \left[\begin{smallmatrix} 3 & 4 \\ 6 & 0 \end{smallmatrix}\right] = 7 \cdot (-24) = -168$$

§4.2, 18 Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{bmatrix}$$

Solution mixing elementary row operations with cofactor expansion: In this case, there are very few zeros, so cofactor expansion isn't very fast. Let's illustrate the row-reduction method. Adding the first row to the second, we get

$$L_{21}(1)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}$$

We could continue with the row-reduction method, but in this case we already have a row/column with lots of zeros. Cofactor expansion along the second row gives:

$$\det(L_{21}(1)(A)) = 0 + 0 + (-1)^{2+3} \cdot (-2) \cdot \det \left[\begin{smallmatrix} 1 & -2 \\ 3 & -1 \end{smallmatrix}\right] = 2 \cdot (-1+6) = 10$$

Since $\det L_{21}(1) = 1$, we have

$$\det(L_{21}(1)(A)) = \det L_{21}(1) \det A = 1 \cdot \det A = \det A$$

Thus we have $\det A = 10$.

Solution using only elementary row operations Alternatively, we can use only elementary row operations. The purpose is to use elementary row operations to get to a matrix whose determinant is easy to calculate, and then to use multiplicativity of the determinant to compute the matrix of A.

$$L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & 5 & -7 \end{bmatrix}$$

$$T_{23}L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -7 \\ 0 & 0 & -2 \end{bmatrix}$$

Note that if we want to use the fact that the determinant of upper triangular matrices are just the product of the diagonal entries, then we could essentially stop here, and we would get $\det T_{23} \det L_{31}(-3) \det L_{21}(1) \det A = -10$, so $\det A = 10$. Alternatively, we can continue row reducing...

$$D_3(-1/2)D_2(1/5)T_{23}L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_{12}(2)D_3(-1/2)D_2(1/5)T_{23}L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & 0 & 1/5 \\ 0 & 1 & -7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_{13}(-1/5)L_{12}(2)D_3(-1/2)D_2(1/5)T_{23}L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_{23}(7/5)L_{13}(-1/5)L_{12}(2)D_3(-1/2)D_2(1/5)T_{23}L_{31}(-3)L_{21}(1)A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we find that the determinant of the product of matrices on the left is 1. Let $\det L_{ij}(a) = 1$, by multiplicativity we have

$$\underbrace{\det D_3(-1/2)}_{-1/2}\underbrace{\det D_2(1/5)}_{1/5}\underbrace{\det T_{23}}_{-1}\det A = \det I = 1$$

It follows that $\det A = 10$.

§4.2, 23 Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Proof. We prove this by induction on the size. Let $P(n)$ be the statement: "the determinant of an $n \times n$ upper triangular matrix is the product of its diagonal entries". We wish to prove $P(n)$ for all $n \geq 1$. The base case $P(1)$ is easy. Now suppose that $P(n-1)$ holds (the inductive hypothesis). We now prove $P(n)$. Let A be an $n \times n$ upper triangular matrix. Since the bottom row has only one nonzero entry, in the nth position, cofactor expansion along the bottom row gives

$$\det A = (-1)^{n+n} A_{nn} \det \tilde{A}_{nn} = A_{nn} \det \tilde{A}_{nn}$$

Since $\tilde{A}_{nn}$ is an $(n-1) \times (n-1)$ matrix, by the inductive hypothesis (i.e., by $P(n-1)$), we know that $\det \tilde{A}_{nn}$ is the product of the diagonal entries of $\tilde{A}_{nn}$. However the diagonal entries of $\tilde{A}_{nn}$ are just the diagonal entries of A with the nth entry omitted, so we have

$$\det \tilde{A}_{nn} = \prod_{i=1}^{n-1} A_{ii}$$

It follows that $\det A = A_{nn} \prod_{i=1}^{n-1} A_{ii} = \prod_{i=1}^{n} A_{ii}$, which shows that $\det A$ is the product of its diagonal entries. This proves $P(n)$, and hence by induction we have proved $P(n)$ for all n. □

§4.3, 12 A matrix $Q \in M_n(\mathbb{R})$ is called *orthogonal* if $QQ^t = I$. Prove that if Q is orthogonal, then $\det Q = \pm 1$.

Proof. Taking determinants, we have $\det(QQ^t) = \det(I) = 1$. On the other hand, by the multiplicativity of det, we have $\det(QQ^t) = \det(Q)\det(Q^t)$. By Theorem 4.8 in the book, $\det(Q^t) = \det(Q)$, so we have $\det(Q)\det(Q) = \det(Q)\det(Q^t) = \det(QQ^t) = \det(I) = 1$. In other words, we have $\det(Q)^2 = 1$. Since $\det(Q) \in \mathbb{R}$ and the only real numbers with square 1 are ± 1, it follows that $\det(Q) = \pm 1$. □

Warning. Note that the converse is not true. A matrix of determinant ± 1 need not be orthogonal!

§4.3, 15 Prove that if $A, B \in M_n(F)$ are similar, then $\det(A) = \det(B)$.

Proof. If A, B are similar, then this means that there is an invertible matrix Q such that $A = QBQ^{-1}$. Taking determinants, we get

$$\det A = \det(QBQ^{-1}) = \det Q \cdot \det B \cdot \det(Q^{-1}) = \det Q \cdot \det B \cdot (\det Q)^{-1} = \det B$$

In the last step, we have used that determinants are elements of F, and in F, $xyx^{-1} = xx^{-1}y = 1y = y$.

□

§4.3, 24 Let $A \in M_n(F)$ have the form

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}$$

Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix and t is a variable.

Hints for 24: Call the given matrix A_n. The goal is to show that $\det(A_n + tI_n)$ is given by the formula

$$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \tag{1}$$

To do this, use induction on n. First show that it holds for $n = 1$ (the "base case"). Then show that if it holds for $n-1$, then it holds for n (this part is called the "inductive step"). To prove the inductive step, use cofactor expansion along a row.

Solution. As per the hint, we will use induction on the size of the matrix A. In the base case $n = 1$, from the definition of A, we easily find $\det(A+tI) = a_0 + t$, so the base case is satisfied. Now suppose the formula (1) holds for $n-1$ (the inductive hypothesis). We will prove that it holds for n. Let A be $n \times n$. It turns out the easiest way to proceed is to cofactor expand along the first row. In this case, cofactor expansion gives:

$$\det(A+tI) = (-1)^{1+1} \cdot t \cdot \det \widetilde{(A+tI)}_{11} + (-1)^{1+n} \cdot a_0 \cdot \det \widetilde{(A+tI)}_{1n}$$

The matrix $\widetilde{(A+tI)}_{11}$ is $(n-1) \times (n-1)$ with exactly the same form as $A+tI$. By the inductive hypothesis, it follows that

$$\det\left(\widetilde{(A+tI)}_{11}\right) = t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1$$

The matrix $\widetilde{(A+tI)}_{1n}$ is $(n-1) \times (n-1)$ and lower-triangular, with every diagonal entry equal to -1, so we have

$$\det\left(\widetilde{(A+tI)}_{1n}\right) = (-1)^{n-1}$$

Putting these last two observations together, we find that

$$\det(A+tI) = t(t^{n-1}+a_{n-1}t^{n-2}+\cdots+a_2t+a_1)+(-1)^{1+n}a_0(-1)^{n-1} = t^n+a_{n-1}t^{n-1}+\cdots+a_2t^2+a_1t+a_0$$

as desired.

§4.4, 2c Evaluate the determinant of the matrix $A = \left[\begin{smallmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{smallmatrix}\right]$.

Note: We're starting to use the field of complex numbers. See the relevant parts of Appendix D for the arithmetic properties of complex numbers.

Solution. Since this matrix is 2 by 2, the determinant is

$$\det A = (2+i)(3-i)-(-1+3i)(1-2i) = 6-2i+3i-i^2+1-2i-3i+6i^2 = 6+i-i^2+1-5i+6i^2 = -4i+2$$

§4.4, 3e Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{bmatrix}$$

by cofactor expansion along the third row.

Solution. Cofactor expanding along the third row gives

$$(-1)^{3+1} \cdot 3 \cdot \underbrace{\det \left[\begin{smallmatrix} 1+i & 2 \\ 0 & 1-i \end{smallmatrix}\right]}_{=2} + (-1)^{3+2} \cdot 4i \cdot \underbrace{\det \left[\begin{smallmatrix} 0 & 2 \\ -2i & 1-i \end{smallmatrix}\right]}_{=4i}$$

Thus, $\det A = 6 - 4i \cdot 4i = 6 + 16 = 22$.

Additional Problems (10 points total)

- Let $\mathcal{B}$ denote the set of all ordered bases for $\mathbb{R}^3$. Let $\mathcal{L}(\mathbb{R}^3)$ denote the set of linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$. Let

$$\mu : \mathcal{L}(\mathbb{R}^3) \times \mathcal{B} \times \mathcal{B} \longrightarrow M_3(\mathbb{R})$$

be the map defined by $\mu(f, \alpha, \beta) = [f]_\alpha^\beta$. Fix $\alpha, \beta \in \mathcal{B}$. A central point of the course so far is that the map

$$\begin{array}{rcl} \mu(*, \alpha, \beta) : \mathcal{L}(\mathbb{R}^3) & \longrightarrow & M_3(\mathbb{R}) \\ f & \mapsto & [f]_\alpha^\beta \end{array}$$

Is a bijection (even, an isomorphism of vector spaces). For any fixed choices of $f \in \mathcal{L}(\mathbb{R}^3), \beta \in \mathcal{B}$, we obtain a map

$$\begin{aligned} \mu_{f,*,\beta} : \mathcal{B} &\longrightarrow M_3(\mathbb{R}) \\ \alpha &\mapsto [f]_\alpha^\beta \end{aligned}$$

Similarly, for any choices $f \in \mathcal{L}(\mathbb{R}^3), \alpha \in \mathcal{B}$, we have a map

$$\begin{aligned} \mu_{f,\alpha,*} : \mathcal{B} &\longrightarrow M_3(\mathbb{R}) \\ \beta &\mapsto [f]_\alpha^\beta \end{aligned}$$

Note that $\mu_{f,\alpha,*}$ and $\mu_{f,*,\beta}$ are not linear! (the set $\mathcal{B}$ is not a vector space). In this exercise we will investigate how the matrix of a linear transformation depends on the choice of basis.

(a) (2 points) Show that for any choice of f, α, $\mu_{f,\alpha,*}$ is not onto.

Proof. The key point here is that f is fixed. Given a fixed linear map f, there is no unique matrix for f (different choices of bases give different matrices), but there are certain properties of f which are inherited by *any* matrix for f. One such example is the rank of f. Indeed, for any basis $\beta \in \mathcal{B}$, we have

$$\mu_{f,\alpha,*}(\beta) = [f]_\alpha^\beta = [I]_{\text{std}}^\beta [f]_{\text{std}}^{\text{std}} [I]_\alpha^{\text{std}}$$

Here, the matrices $[I]_{\text{std}}^\beta$ and $[I]_\alpha^{\text{std}}$ are both change of basis matrices, and hence are invertible, so it follows that $\text{rank}(f) = \text{rank}[f]_\alpha^\beta$ for any choices of bases $\alpha, \beta \in \mathcal{B}$. Thus, for fixed f, every matrix in the image of $\mu_{f,\alpha,*}$ must have the same rank as f. Since $M_3(\mathbb{R})$ has matrices of every rank between 0 and 3, it follows that $\mu_{f,\alpha,*}$ cannot be onto.

To be concrete, if f is invertible (i.e., rank 3), then the image of $\mu_{f,\alpha,*}$ consists only of invertible matrices (i.e., rank 3). Thus, for example, in this case the zero matrix does not lie in the image of $\mu_{f,\alpha,*}$. If f is rank 2 or 1, then again the zero matrix does not lie in the image of $\mu_{f,\alpha,*}$. If f is rank 0, then the identity matrix does not lie in the image of $\mu_{f,\alpha,*}$. $\square$

(b) (2 points) Show that for any choice of f, β, $\mu_{f,*,\beta}$ is not onto.

Proof. For the same reason as in part (a), any matrix in the image of $\mu_{f,*,\beta}$ has the same rank as f. Since $M_3(\mathbb{R})$ has matrices of any rank (between 0 and 3), it follows that $\mu_{f,*,\beta}$ cannot be onto. $\square$

(c) (3 points) Let $\alpha \in \mathcal{B}$. Show that $\mu_{f,\alpha,*}$ is 1-1 if and only if f is invertible.

Hint: For (c) and (d), for the direction invertible $\Longrightarrow$ 1-1, you can use the fact, proven on the quiz, that $[I]_\beta^\gamma = I_n \iff \beta = \gamma$, together with the identity $[fg]_\alpha^\gamma = [f]_\beta^\gamma [g]_\alpha^\beta$. The other direction is more subtle.

Proof. Suppose f is invertible, and suppose $\mu_{f,\alpha,*}(\beta) = \mu_{f,\alpha,*}(\beta')$, then $[f]_\alpha^\beta = [f]_\alpha^{\beta'}$, but this means

$$[f]_\alpha^\beta = [f]_\alpha^{\beta'} = [I]_\beta^{\beta'} [f]_\alpha^\beta$$

Since f is invertible, so is $[f]_\alpha^\beta$, so canceling $[f]_\alpha^\beta$ (i.e., right-multiplying by its inverse), we get $I_3 = [I]_\beta^{\beta'}$, where I_3 denotes the 3×3 identity matrix. From problem 1 on Quiz 6, this implies that $\beta = \beta'$. Thus we've shown that $\mu_{f,\alpha,*}(\beta) = \mu_{f,\alpha,*}(\beta')$ implies that $\beta = \beta'$. In other words, $\mu_{f,\alpha,*}$ is 1-1.

Next we will show that if f is not invertible, then $\mu_{f,\alpha,*}$ is not 1-1. If f is not invertible, by the rank-nullity theorem, it has rank ≤ 2, so f is not onto. Let $\{\beta_1, \beta_2\}$ be a basis for the image of f. By Corollary 2 (in §1.6, p48 in the book), we can extend $\{\beta_1, \beta_2\}$ to a basis for $\mathbb{R}^3$. Let β_3 be the third member of such an extension. Note that such extensions are never unique! For example, if $\{\beta_1, \beta_2, \beta_3\}$ is one basis of $\mathbb{R}^3$ extending $\{\beta_1, \beta_2\}$, then $\{\beta_1, \beta_2, 2\beta_3\}$ is another basis extending $\{\beta_1, \beta_2\}$. Thus, let $\beta_3' \neq \beta_3$ be another extension. Let $\beta := \{\beta_1, \beta_2, \beta_3\}$, and let $\beta' := \{\beta_1, \beta_2, \beta_3'\}$. We claim that $[f]_\alpha^\beta = [f]_\alpha^{\beta'}$. Indeed, the ith column of $[f]_\alpha^\beta$ is the coordinate vector $[f(\alpha_i)]_\beta$, and similarly the ith column of $[f]_\alpha^{\beta'}$ is the coordinate vector $[f(\alpha_i)]_{\beta'}$. Since $f(\alpha_i) \in \text{Span}\{\beta_1, \beta_2\}$ (for any $i = 1, 2, 3$), the coordinate vectors $[f(\alpha_i)]_\beta$ and $[f(\alpha_i)]_{\beta'}$ are identical (they both have the form $(*, *, 0)$. It follows that the matrices $[f]_\alpha^\beta$ and $[f]_\alpha^{\beta'}$ are equal. This shows that $\mu_{f,\alpha,*}(\beta) = \mu_{f,\alpha,*}(\beta')$ even though $\beta \neq \beta'$. It follows that $\mu_{f,\alpha,*}$ is not 1-1.

$\square$

(d) (3 points) Let $\beta \in \mathcal{B}$. Show that $\mu_{f,*,\beta}$ is 1-1 if and only if f is invertible.

Proof. Suppose f is invertible. We will show that $\mu_{f,*,\beta}$ is 1-1. This proceeds similarly to part (c). Suppose $\alpha, \alpha' \in \mathcal{B}$ are bases with $\mu_{f,\alpha,\beta} = \mu_{f,\alpha',\beta}$, then we have

$$[f]_\alpha^\beta = [f]_{\alpha'}^\beta = [f]_\alpha^\beta [I]_{\alpha'}^\alpha$$

Since f is invertible, so is $[f]_\alpha^\beta$, so cancelling them (i.e., left multiplying by its inverse), we find again that

$$I_3 = [I]_{\alpha'}^\alpha$$

By Problem 1 on Quiz 6, this implies that $\alpha = \alpha'$. Since α, α' were arbitrary, this shows that $\mu_{f,*,\beta}$ is 1-1.

Now suppose f is not invertible. We must show that $\mu_{f,*,\beta}$ is not 1-1. The proof uses a similar idea, but now we will adjust the basis for the domain (instead of the codomain). Since f is not invertible, by the rank-nullity theorem it has nullity > 0. In particular, there is a nonzero vector $v \in N(f)$. Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} \in \mathcal{B}$ be any basis of $\mathbb{R}^3$. Let $\alpha' := \{\alpha_1 + v, \alpha_2, \alpha_3\}$. Thus $\alpha_1' = \alpha_1 + v$, and $\alpha_i' = \alpha_i$ for $i = 2, 3$. Then we claim that $[f]_\alpha^\beta = [f]_{\alpha'}^\beta$. Since the ith columns of $[f]_\alpha^\beta$ and $[f]_{\alpha'}^\beta$ are $[f(\alpha_i)]_\beta$ and $[f(\alpha_i')]_\beta$ respectively, it follows easily that the 2nd and 3rd columns of $[f]_\alpha^\beta, [f]_{\alpha'}^\beta$ are identical. On the other hand, the first column of $[f]_{\alpha'}^\beta$ is

$$[f(\alpha_1')]_\beta = [f(\alpha_1 + v)]_\beta = [f(\alpha_1) + f(v)]_\beta = [f(\alpha_1) + 0]_\beta = [f(\alpha_1)]_\beta$$

so the first columns are also identical. Thus, the matrices $[f]_\alpha^\beta, [f]_{\alpha'}^\beta$ are the same. This shows that $\mu_{f,*,\beta}$ is not 1-1, as desired.

Remark. In general, if α' is any basis such that $\alpha_i' - \alpha_i \in N(f)$ for each i, then $[f]_\alpha^\beta = [f]_{\alpha'}^\beta$. Can you prove this? $\square$