MATH 350 Linear Algebra
Homework 6

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October 28, 2022

Problems

Book Problems 2 points each, 20 points total

- §4.2, Problems 7, 8, 14, 18, 23

  For 18, use elementary row operations.

- §4.3, Problems 12, 15, 24

- Hints for 24: Call the given matrix A_n. The goal is to show that $\det(A_n + tI_n)$ is given by the formula $t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. To do this, use induction on n. First show that it holds for $n = 1$ (the "base case"). Then show that if it holds for $n - 1$, then it holds for n (this part is called the "inductive step"). To prove the inductive step, use cofactor expansion along a row.

- §4.4, Problems 2(c), 3(e)

  In problem 3(e), you're asked to evaluate the determinant by cofactor expansion along the third row.

  Note: We're starting to use the field of complex numbers. See the relevant parts of Appendix D for the arithmetic properties of complex numbers.

Additional Problems (10 points total)

- Let $\mathcal{B}$ denote the set of all ordered bases for $\mathbb{R}^3$. Let $\mathcal{L}(\mathbb{R}^3)$ denote the set of linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$. Let

  $$\mu : \mathcal{L}(\mathbb{R}^3) \times \mathcal{B} \times \mathcal{B} \longrightarrow M_3(\mathbb{R})$$

  be the map defined by $\mu(f, \alpha, \beta) = [f]_\alpha^\beta$. Fix $\alpha, \beta \in \mathcal{B}$. A central point of the course so far is that the map

  $$\begin{array}{rcl} \mu(*, \alpha, \beta) : \mathcal{L}(\mathbb{R}^3) & \longrightarrow & M_3(\mathbb{R}) \\ f & \mapsto & [f]_\alpha^\beta \end{array}$$

  Is a bijection (even, an isomorphism of vector spaces). For any fixed choices of $f \in \mathcal{L}(\mathbb{R}^3), \beta \in \mathcal{B}$, we obtain a map

  $$\begin{array}{rcl} \mu_{f,*,\beta} : \mathcal{B} & \longrightarrow & M_3(\mathbb{R}) \\ \alpha & \mapsto & [f]_\alpha^\beta \end{array}$$

  Similarly, for any choices $f \in \mathcal{L}(\mathbb{R}^3), \alpha \in \mathcal{B}$, we have a map

  $$\begin{array}{rcl} \mu_{f,\alpha,*} : \mathcal{B} & \longrightarrow & M_3(\mathbb{R}) \\ \beta & \mapsto & [f]_\alpha^\beta \end{array}$$

  Note that $\mu_{f,\alpha,*}$ and $\mu_{f,*,\beta}$ are not linear! (the set $\mathcal{B}$ is not a vector space). In this exercise we will investigate how the matrix of a linear transformation depends on the choice of basis.

(a) (2 points) Show that for any choice of f, α, $\mu_{f,\alpha,*}$ is not onto.

(b) (2 points) Show that for any choice of f, β, $\mu_{f,*,\beta}$ is not onto.

(c) (3 points) Let $\alpha \in \mathcal{B}$. Show that $\mu_{f,\alpha,*}$ is 1-1 if and only if f is invertible.

(d) (3 points) Let $\beta \in \mathcal{B}$. Show that $\mu_{f,*,\beta}$ is 1-1 if and only if f is invertible.

Hint: For (c) and (d), for the direction invertible $\implies$ 1-1, you can use the fact, proven on the quiz, that $[I]_\beta^\gamma = I_n \iff \beta = \gamma$, together with the identity $[fg]_\alpha^\gamma = [f]_\beta^\gamma [g]_\alpha^\beta$. The other direction is more subtle.