MATH 350 Linear Algebra
Homework 5 Solutions

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Problems

Book Problems 2 points each, 26 points total

- §2.5, Problems 2(b), 2(c), 3(c), 5, 6(b), 6(c), 7(a)
- §3.2, Problems 5(e), 6(a), 7
- §3.3, Problems 2(b), 3(b)
- §3.4, Problem 6

Additional Problems (4 points total)

- A 3×3 magic square is a 3×3 matrix with entries in $\mathbb{R}$ where every column, row, and diagonal sums to 0 (there are 2 diagonals).
  - (2 points) Show that the set of 3×3 magic squares is a subspace of $M_{3\times 3}(\mathbb{R})$.
  - (2 points) Compute the dimension of the space of 3×3 magic squares.

Solutions

In my solutions, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

In general, solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

§2.5,2(b) For $\beta = \{(-1,3),(2,-1)\}$ and $\beta' = \{(0,10),(5,0)\}$ for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β coordinates.

Solution. We seek to find $[I]_{\beta'}^{\beta}$. By the definition of the matrix associated to a linear transformation, the 1st column of this matrix is the coordinate vector $[\beta_1']_\beta$. To calculate this, we want to solve the system

$$a_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

The solution is: $a_1 = 4, a_2 = 2$, so the first column of $[I]_{\beta'}^{\beta}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Similarly, the second column is the coordinate vector $[\beta_2']_\beta$. Since

$$1 \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

we find that the second column is $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$. Thus

$$[I]_{\beta'}^{\beta} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

§2.5,2(c) For $\beta = \{(2,5),(-1,-3)\}$ and $\beta' = \{e_1, e_2\}$ for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β coordinates.

Solution. As in part (b), the columns of $[I]_{\beta'}^{\beta}$ are the coordinate vectors of β_1', β_2' with respect to β. Solving the necessary equations, we find that

$$3 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 5 \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} = e_1$$

we find that the first column is $\left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]$. Since

$$-1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + (-2) \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} = e_2$$

we find that the second column is $\left[\begin{smallmatrix}-1\\-2\end{smallmatrix}\right]$. Thus, the change of basis matrix is

$$[I]_{\beta'}^{\beta} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$$

§2.5,3(c) Let $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$ be bases of $P_2(\mathbb{R})$. Find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

Solution. Again, we want to find $[I]_{\beta'}^{\beta}$. The first column is the coordinate vector $[\beta_1']_\beta$, so we want to express $\beta_1' = 1$ as a linear combination of the vectors in β. In other words, we want to find a_1, a_2, a_3 such that

$$a_1(2x^2 - x) + a_2(3x^2 + 1) + a_3 x^2 = 1$$

Collecting terms of the same degree, solving for a_1, a_2, a_3 is equivalent to solving the linear system

$$\begin{aligned} a_2 &= 1 \\ -a_1 &= 0 \\ 2a_1 + 3a_2 + a_3 &= 0 \end{aligned}$$

Thus we have $a_1 = 0$, $a_2 = 1$, $a_3 = -3$. This gives the first column of $[I]_{\beta'}^{\beta}$. For the second column, we want to write $x = \beta_2'$ as a linear combination of the vectors in β. This is equivalent to the linear system

$$\begin{aligned} a_2 &= 0 \\ -a_1 &= 1 \\ 2a_1 + 3a_2 + a_3 &= 0 \end{aligned}$$

Thus we have $a_1 = -1$, $a_2 = 0$, $a_3 = 2$. This gives the second column of $[I]_{\beta'}^{\beta}$. For the third column, we want to write $x^2 = \beta_3'$ as a linear combination of the vectors in β. This is equivalent to the linear system

$$\begin{aligned} a_2 &= 0 \\ -a_1 &= 0 \\ 2a_1 + 3a_2 + a_3 &= 1 \end{aligned}$$

Thus we have $a_1 = a_2 = 0$ and $a_3 = 1$. Thus, we have

$$[I]_{\beta'}^{\beta} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

§2.5,5 Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1+x, 1-x\}$. Use Theorem 2.23 and the fact that $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{smallmatrix}\right]$ to find $[T]_{\beta'}$.

Solution. By Theorem 2.23, we have

$$[T]_{\beta'} = ([I]_{\beta'}^{\beta})^{-1}[T]_\beta[I]_{\beta'}^{\beta} \tag{1}$$

First we find $[I]_{\beta'}^{\beta}$. The first column is just $[1+x]_\beta = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and the second column is $[1-x]_\beta = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. Thus $[I]_{\beta'}^{\beta} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

Next we find $[T]_\beta$. Its first column is $[T(1)]_\beta = [0]_\beta = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, and its second column is $[T(x)]_\beta = [1]_\beta = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Thus $[T]_\beta = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

It follows from (1) that

$$[T]_{\beta'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

At this point we should check our work: By definition, the first column of $[T]_{\beta'}$ is $[T(\beta_1')]_{\beta'} = [\frac{d}{dx}(1+x)]_{\beta'} = [1]_{\beta'}$. Since $1 = \frac{1}{2}(1+x)+\frac{1}{2}(1-x)$, the first column is correct. Similarly, the second column is by definition $[T(\beta_2')]_{\beta'} = [\frac{d}{dx}(1-x)]_{\beta'} = [-1]_{\beta'}$. Since $-1 = \frac{-1}{2}(1+x) + \frac{-1}{2}(1-x)$, the second column is also correct.

§2.5,6(b) For $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ and $\beta = \left\{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right\}$, find $[L_A]_\beta$. Also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

Solution. If std denotes the standard basis of $\mathbb{R}^2$, then $[L_A]_{\text{std}} = A$. However in our case $\beta \neq \text{std}$, so we must be more careful. By definition, the first column of $[L_A]_\beta$ is the coordinate vector $[L_A(\beta_1)]_\beta = [(3,3)]_\beta$. Thus we want to express $(3,3)$ in terms of the vectors in β. Clearly $(3,3) = 3 \cdot \beta_1 + 0 \cdot \beta_2$, so the first column of $[L_A]_\beta$ is $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$.

Similarly the second column of $[L_A]_\beta$ is $[L_A(\beta_2)]_\beta = [(-1,1)]_\beta$. Solving the linear system

$$a_1\beta_1 + a_2\beta_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

yields $a_1 = 0, a_2 = -1$. Thus the second column of $[L_A]_\beta$ is $\left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$, so we have

$$[L_A]_\beta = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

To find Q, one can use the identity $[fg]_\alpha^\gamma = [f]_\beta^\gamma[g]_\alpha^\beta$. This identity implies that if $I : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the identity, then

$$[L_A]_\beta = [L_A]_\beta^\beta = [I \circ L_A \circ I]_\beta^\beta = [I]_{\text{std}}^\beta[L_A]_{\text{std}}^{\text{std}}[I]_\beta^{\text{std}} = [I]_{\text{std}}^\beta A[I]_\beta^{\text{std}}$$

Since we have $([f]_\alpha^\beta)^{-1} = [f^{-1}]_\beta^\alpha$ (for any invertible linear map f and bases α, β), it follows that we can take $Q = [I]_\beta^{\text{std}}$. Its first column is $[\beta_1]_{\text{std}} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, and its second column is $[\beta_2]_{\text{std}} = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, so we can take $Q = [I]_\beta^{\text{std}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

At this point you should check by multiplying matrices that indeed $[L_A]_\beta = Q^{-1}AQ$.

§2.5,6(c) Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Find $[L_A]_\beta$. Also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

Solution. This is similar to part (b). The first column of $[L_A]_\beta$ is $[L_A(\beta_1)]_\beta = [(1,3,2)]_\beta$. Since

$$2\beta_1 - 2\beta_2 + \beta_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix},$$

this first column is $(2,-2,1)$. The second column is $[L_A(\beta_2)]_\beta = [(0,3,1)]_\beta$. Since

$$2\beta_1 - 3\beta_2 + \beta_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$

the second column is $(2,-3,1)$. The third column is $[L_A(\beta_3)]_\beta = [(0,4,2)]_\beta$. Since

$$2\beta_1 - 4\beta_2 + 2\beta_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},$$

the third column is $(2,-4,2)$. Thus

$$[L_A]_\beta = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{bmatrix}$$

As in part (b), we can take $Q = [I]_\beta^{\text{std}}$, which we easily compute is

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

§2.5,7(a) In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x,y)$ is the reflection of the point (x,y) about the line L.

Solution. The first step is to find a good basis with respect to which T is easily understandable. As done in class, let us take a point on L, and a point which is "perpendicular" to L. For example, we can take $\beta = \{(1,m),(-m,1)\}$. Then $T(\beta_1) = \beta_1$ and $T(\beta_2) = -\beta_2$, so we have

$$[T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To find an expression for T, we need to find $[T]_{\text{std}}$. By Theorem 2.23, we have

$$[T]_{\text{std}} = [I]_\beta^{\text{std}} [T]_\beta [I]_{\text{std}}^\beta$$

(Remember that $[T]_\beta$ is shorthand for $[T]_\beta^\beta$) Since $[I]_\beta^{\text{std}} = ([I]_{\text{std}}^\beta)^{-1}$, it suffices to compute one of $[I]_\beta^{\text{std}}$ or $[I]_{\text{std}}^\beta$ (since the other is just the inverse). In this case it is easier to compute $[I]_\beta^{\text{std}}$, which is just

$$[I]_\beta^{\text{std}} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$$

It follows that we have

$$[T]_{\text{std}} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix}$$

To check that this matrix is correct, you should verify that $[T]_{\text{std}} \left[\begin{smallmatrix} 1 \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ m \end{smallmatrix}\right]$ and $[T]_{\text{std}} \left[\begin{smallmatrix} -m \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} m \\ -1 \end{smallmatrix}\right]$.

To complete the problem, one should really find a formula for $T(x,y)$. In this case, we have

$$T(x,y) = [T]_{\text{std}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note that you should not substitute mx for y in this formula. This is because x,y should represent an arbitrary vector in $\mathbb{R}^2$, not just a vector on the line $y = mx$.

§3.2,5(e) Compute the rank and inverse (if it exists) of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution. Use Gaussian elimination on the augmented matrix $(A|I)$ (see p160-163 in the book). Here are some intermediate steps:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 3 & 3 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 3 & -1/2 & 1 & 3/2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right]$$

It follows that the final matrix on the right is the inverse, namely:

$$\begin{bmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{bmatrix}$$

§3.2,6(a) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = f''(x) + 2f'(x) - f(x)$$

Determine if T is invertible, and compute T^{-1} if it exists.

Solution. We will work with respect to the standard basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$. Since $T(1) = -1$ and $T(x) = 2 - x$ and $T(x^2) = 2 + 4x - x^2$, we have

$$[T] = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

We can use the same method as in 5(e) to calculate the inverse:

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

This shows that

$$[T]^{-1} = [T^{-1}] = \begin{bmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

But this means that

$$\begin{aligned} T^{-1}(1) &= -1 \\ T^{-1}(x) &= -x - 2 \\ T^{-1}(x^2) &= -x^2 - 4x - 10 \end{aligned}$$

In a formula, we can say:

$$T^{-1}(a + bx + cx^2) = -a - bx - 2b - cx^2 - 4cx - 10c = -cx^2 + (-b - 4c)x + (-10c - 2b - a)$$

§3.2,7 Express the invertible matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

as a product of elementary matrices.

Solution. Recall that if E is an elementary matrix then for any matrix B, EB is the matrix obtained by applying to B the elementary row operation associated to E. Thus to express A as a product of elementary matrices, it suffices to keep track of the elementary row operations obtained while row-reducing A.

We will use the following notation: For $1 \le i < j \le 3$, let T_{ij} be the elementary matrix associated to swapping rows i and j. For $a \in F$, $a \neq 0$, let $D_i(a)$ be the elementary matrix associated to scaling row i by a. For $1 \le i, j \le 3$ and $a \in F$, let $L_{ij}(a)$ be the elementary matrix associated to adding a times row j to row i.

Since there are many ways to row reduce a matrix, the expression of A in terms of elementary matrices is not unique. Here we give one method:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \overset{L_{21}(-1)}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \overset{L_{31}(-1)}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \overset{D_2(-1/2)}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\overset{L_{32}(1)}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{L_{12}(-2)}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{L_{13}(-1)}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

This shows that $I = L_{13}(-1)L_{12}(-2)L_{32}(1)D_2(-1/2)L_{31}(-1)L_{21}(-1)A$. Thus,

$$A = L_{21}(-1)^{-1}L_{31}(-1)^{-1}D_2(-1/2)^{-1}L_{32}(1)^{-1}L_{12}(-2)^{-1}L_{13}(-1)^{-1}$$

Since inverses of elementary matrices are elementary, this expresses A as a product of elementary matrices. Again, note that this expression is not unique.

§3.3,2(b) Find the dimension of and a basis for the solution space to the linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 4x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Solution. This homogeneous system can be expressed as the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & -2 \end{bmatrix}}_{A:=} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The reduced row echelon form of A is

$$A' = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \end{bmatrix}$$

It follows that $\text{rank}(A) = 2$, so $\text{nullity}(A) = 1$. Since elementary row operations do not change the null space, we find that $N(A) = N(A')$. Since A' is in reduced row echelon form, its null space is especially easy to see: We must have $x_1 + \frac{-1}{3}x_3 = 0 = x_2 + \frac{-2}{3}x_3$, where x_3 is the free variable. Setting x_3 conveniently to be $x_3 = 3$ gives the nontrivial solution $(1, 2, 3)$. Since $\text{nullity}(A) = 1$, $(1, 2, 3)$ is a basis for the solution space.

§3.3,3(b) Using the results of §3.3, 2(b), find all solutions to the system

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 4x_1 + x_2 - 2x_3 &= 3 \end{aligned}$$

Solution. Row reducing the augmented matrix, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 4 & 1 & -2 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right]$$

Thus our given system is equivalent to the system

$$\begin{aligned} x_1 - \frac{1}{3}x_3 &= 2/3 \\ x_2 - \frac{2}{3}x_3 &= 1/3 \end{aligned}$$

One solution is easily found by setting $x_3 = 0$, in which case we get $x_1 = 2/3$, $x_2 = 1/3$, thus one solution to the system is $(2/3, 1/3, 0)$. By 2(b), the associated homogeneous system has a 1-dimensional space of solutions, with basis $(1, 2, 3)$. Thus by Theorem 3.9, the space of all solutions to the nonhomogeneous system is:

$$(2/3, 1/3, 0) + \mathrm{Span}\{(1, 2, 3)\}$$

§3.4,6 Let the reduced row-echelon form of A be

$$B = \begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $a_1, \ldots, a_6$ denote the columns of A. Determine A if

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix} \qquad a_6 = \begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix}$$

Solution. Since B is the reduced row-echelon form of A, we have that $B = EA$, where E is a product of elementary matrices (in particular, E is invertible). As described in the hint from the email, if X is any $m \times n$ matrix and U an invertible $m \times m$ matrix, then any linear relation between the columns of X are also satisfied by the columns of UX. This can be seen by examining how the multiplication UX is defined. Similarly, multiplying by U^{-1}, and linear relations between the columns of UX are also satisfied by the columns of X.

Let $b_1, \ldots, b_6$ denote the columns of b. It's easily verified that we have

$$\begin{aligned} b_2 &= -3b_1 \\ b_4 &= 4b_1 + 3b_3 \\ b_6 &= 5b_1 + 2b_3 - b_5 \end{aligned}$$

It follows that the same relations hold if the b_i's are replaced by a_i's. In particular, the first tells us that $a_2 = -3a_1$. The second tells us that $a_4 = 4a_1 + 3a_3$, and the third tells us that $a_5 = 5a_1 + 2a_3 - a_6$. Since a_1, a_3, a_6 are given to us, we now know what A is, namely:

$$A = \begin{bmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{bmatrix}$$

At this point, we should check that the reduced row-echelon form of A is indeed B.

(A1) A 3×3 magic square is a 3×3 matrix with entries in $\mathbb{R}$ where every column, row, and diagonal sums to 0.

- Show that the set of 3×3 magic squares is a subspace of $M_{3\times 3}(\mathbb{R})$

  Solution. A (3×3) magic square can be viewed as a 9-tuple of numbers:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

  The condition that the first row sums to 0 is the same as requiring:

$$x_1 + x_2 + x_3 = 0$$

  The condition that the first column sums to 0 is the same as requiring:

$$x_1 + x_4 + x_7 = 0$$

  The condition that the diagonals sum to 0 is the same as requiring:

$$x_1 + x_5 + x_9 = x_3 + x_5 + x_7 = 0$$

  In particular, a 9-tuple of numbers $(x_1, \ldots, x_9)$ is a magic square if and only if it satisfies a certain homogeneous linear system consisting of 8 equations (3 for rows, 3 for columns, 2 for diagonals). Alternatively, the set of magic squares is precisely the null space of the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

  Since the null space is a subspace, it follows that the set of magic squares is a subspace of $M_{3\times 3}(\mathbb{R})$.

- Compute the dimension of the space of 3×3 magic squares.

  Solution. As described above, the space of magic squares is just the null space of M. Thus we must compute the nullity of M. This can be done by finding the reduced row echelon form of M. Doing this, we find that $\text{rank}(M) = 7$, so $\text{nullity}(M) = 2$.