MATH 350 Linear Algebra
Homework 4

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October 18, 2022

Homework 3 Problems

Book Problems 2 points each, 30 points total

- §2.2, Problems 4, 5(d)
- §2.3, Problems 3(a), 3(b), 4(d), 12(a), 12(b), 12(c)
  Note that problem 4(d) in §2.3 refers to Problem 5(d) in §2.2.
- §2.4, Problems 4, 7(a), 7(b), 24(a), 24(b), 24(c), 24(d)
  Hint: In problems 4 and 7, use the correspondence between linear transformations and matrices.

Solutions

In my solutions, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

In general, solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

§2.2, 4 Define $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = (a+b)+(2d)x+bx^2$. Let $\beta = \{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}$ and $\gamma = \{1, x, x^2\}$. Compute $[T]_\beta^\gamma$.

Solution. Recall, that the ith column of $[T]_\beta^\gamma$ is just the image of the ith basis vector of β, expressed as the coordinate vector with respect to γ. Thus if $\beta_1, \ldots, \beta_4$ are the vectors in β, then for example we have

$$\begin{aligned} T(\beta_1) &= 1 \\ T(\beta_2) &= 1 + x^2 \\ T(\beta_3) &= 0 \\ T(\beta_4) &= 2x \end{aligned}$$

Thus

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

§2.2, 5(d) Let $\alpha = \{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}$, $\beta = \{1, x, x^2\}$, and $\gamma = \{1\}$. Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

Solution. Letting $\beta_1, \beta_2, \beta_3$ be the vectors in β, we have $T(\beta_1) = 1, T(\beta_2) = 2, T(\beta_3) = 4$, so $[T]_\beta^\gamma$ is the 1x3 matrix $[1\ \ 2\ \ 4]$.

§2.3, 3(a) Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \qquad U(a + bx + cx^2) = (a + b, c, a - b)$$

Let $\beta = \{1, x, x^2\}, \gamma = \{e_1, e_2, e_3\}$ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively. Compute $[U]_\beta^\gamma, [T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

Solution. We have

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad [T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix} \qquad [UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$$

As predicted by Theorem 2.11, we have $[UT]_\beta^\gamma = [U]_\beta^\gamma[T]_\beta$.

§2.3, 3(b) With notation as above, let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from part (a) and Theorem 2.14 to verify your result.

Solution. Note that $h(x) \in P_2(\mathbb{R})$, so $[h(x)]_\beta$ is the coordinate vector of $h(x)$ with respect to the basis β. Thus we have

$$[h(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Since $U(h(x)) = (1, 1, 5)$, we have

$$[U(h(x))]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

As predicted by Theorem 2.14, we can check that indeed $[U(h(x))]_\gamma = [U]_\beta^\gamma[h(x)]_\beta$.

§2.3, 4(d) Let $T : P_2(\mathbb{R}) \to \mathbb{R}$ be the linear transformation $T(f(x)) = f(2)$, let $\beta = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R})$, and let $\gamma = \{1\}$ be a basis of $\mathbb{R}$. Use Theorem 2.14 to compute the vector $[T(f(x))]_\gamma$, where $f(x) = 6 - x + 2x^2$.

Solution. First, $[f(x)]_\beta$ is the column vector $(6, -1, 2)$. By Theorem 2.14, $[T(f(x))]_\gamma$ is the 1×1 matrix

$$[T(f(x))]_\gamma = [T]_\beta^\gamma[f(x)]_\beta = [1 \;\; 2 \;\; 4] \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} = [1 \cdot 6 + 2 \cdot (-1) + 4 \cdot 2] = [12]$$

This can also be checked directly, since $T(f(x)) = f(2) = 12$.

§2.3, 12(a) Let V, W, Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear. Prove that if UT is 1-1, then T is 1-1. Must U also be 1-1?

Proof. Suppose T is not 1-1. Then there exist $v \neq v' \in V$ such that $T(v) = T(v')$, but then $(UT)(v) = U(T(v)) = U(T(v')) = (UT)(v')$, so this shows that UT is not 1-1. By the contrapositive, it follows that if UT is 1-1, then T must also be 1-1.

Even if UT is 1-1, U need not be 1-1. For example, take $T : \mathbb{R} \to \mathbb{R}^2$ be given by $x \mapsto (x, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $(x, y) \mapsto (x, 0, 0)$. Then clearly $UT : \mathbb{R} \to \mathbb{R}^3$ is just the map $x \mapsto (x, 0, 0)$ which is clearly 1-1. On the other hand, U is clearly not 1-1, since its kernel is the y-axis inside $\mathbb{R}^2$.

□

§2.3, 12(b) With notation as above, prove that if UT is onto, then U is onto. Must T also be onto?

Proof. Suppose UT is onto. Then for any $z \in Z$, there is a $v \in V$ such that $(UT)(v) = z$. But that means that $U(T(v)) = (UT)(v) = z$, so U maps $T(v)$ to z. This shows that U is also onto.

Even if UT is onto, T need not be onto. For example, take $T : \mathbb{R} \to \mathbb{R}^2$ be given by $x \mapsto (x, 0)$, and $U : \mathbb{R}^2 \to \mathbb{R}$ be given by $(x, y) \mapsto x$. Then $UT : \mathbb{R} \to \mathbb{R}$ is the identity map $x \mapsto x$, but obviously T is not onto. □

§2.3, 12(c) With notation as above, prove that if U and T are 1-1 and onto, then UT is also 1-1 and onto.

Proof. Suppose U and T are both 1-1 and onto (i.e., they are both isomorphisms). We will first show that UT is 1-1, and then show that UT is onto. If $(UT)(v) = U(T(v)) = 0$, then since U is 1-1, $T(v) = 0$, and since T is 1-1, $v = 0$. This shows that UT is 1-1. For any $z \in Z$, since U is onto, there is a $w \in W$ such that $U(w) = z$. Since T is onto, there is a $v \in V$ such that $T(v) = w$. Then $(UT)(v) = U(T(v)) = U(w) = z$. This shows that UT is onto. □

§2.4, 4 Let A, B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. We have $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$. Similarly, we have $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. This shows that $B^{-1}A^{-1}$ is the inverse of AB. Note that $B^{-1}A^{-1} \neq A^{-1}B^{-1}$ in general! □

§2.4, 7(a) Let A be an $n \times n$ matrix. Suppose that $A^2 = O$. Prove that A is not invertible.

Proof. Recall that O is the zero matrix. If A is invertible, then we may multiply $A^2 = O$ on both sides by A^{-1} to get $A^{-1}A^2 = A^{-1}O = O$. But this implies $A = A^{-1}A^2 = O$, but clearly O is not invertible, so this contradicts the invertibility of A. Thus A could not have been invertible.

□

§2.4, 7(b) Let A be an $n \times n$ matrix. Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

Proof. If A were invertible, then $A^{-1}AB = A^{-1}O = O$, so $B = A^{-1}AB = O$. Since B is assumed nonzero, this means that A could not be invertible.

Remark. You should think of this as being analogous to the statement: If $a \in \mathbb{R}$ is invertible (i.e., nonzero), then the only way for $ab = 0$ is that $b = 0$. □

§2.4, 24(a) Let V and Z be vector spaces and let $T : V \to Z$ be a linear transformation that is onto. Define the mapping

$$\overline{T} : V/N(T) \to Z \qquad \text{by} \qquad \overline{T}(v + N(T)) = T(v)$$

for any coset $v + N(T) \in V/N(T)$. Prove that $\overline{T}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $T(v) = T(v')$.

Proof. Suppose that $v + N(T) = v' + N(T)$. Recall, by an earlier homework problem (§1.3, 31(b)) that this happens if and only if $v - v' \in N(T)$. Then $T(v) - T(v') = T(v - v') = 0$, so $T(v) = T(v')$ as desired. □

§2.4, 24(b) Prove that $\overline{T}$ is linear.

Proof. Let $v + N(T), v' + N(T) \in V/N(T)$. Then $\overline{T}((v + N(T)) + (v' + N(T))) = \overline{T}((v + v') + N(T)) = T(v + v') = T(v) + T(v') = \overline{T}(v + N(T)) + \overline{T}(v' + N(T))$. The key step here is the second to last equality, using the linearity of T. This shows that $\overline{T}$ is additive.

Let $v + N(T)$ and $a \in F$. Then $\overline{T}(a(v + N(T))) = \overline{T}(av + N(T)) = T(av) = aT(v) = a\overline{T}(v + N(T))$. Again, the key step is to use the linearity of T. This shows that $\overline{T}$ respects scalar multiplication. Thus, $\overline{T}$ is linear. □

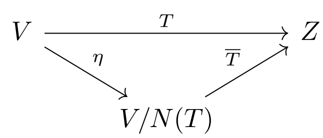
§2.4, 24(c) Prove that $\overline{T}$ is an isomorphism.

Proof. Since $\overline{T}$ is linear, it remains to check that it is bijective (i.e., 1-1 and onto). Note that $\overline{T}(v+N(T)) = T(v) = 0$ if and only if $v \in N(T)$ if and only if $v + N(T)$ is the zero vector in $V/N(T)$. This shows that $\overline{T}$ is 1-1.

Now let $z \in Z$. Since T is onto, there is some $v \in V$ such that $T(v) = z$. But then $\overline{T}(v+N(T)) = T(v) = z$, so $\overline{T}$ is onto, as desired.

Remark. One might be tempted to use the rank-nullity theorem here, but this would not be valid since we have not assumed that V, Z are finite dimensional. □

§2.4, 24(d) Prove that the diagram



commutes. That is, prove that $T = \overline{T}\eta$.

Proof. Recall from Exercise 42 of §2.1 that $\eta : V \to V/N(T)$ is the map sending $v \mapsto v + N(T)$. Then unfolding definitions, for any $v \in V$ we have

$$(\overline{T}\eta)(v) = \overline{T}(\eta(v)) = \overline{T}(v + N(T)) = T(v)$$

This shows that $\overline{T}\eta = T$ as desired. □