MATH 350 Linear Algebra
Homework 3 Solutions

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Homework 3 Problems

Book Problems

- (12 points) §1.6, Problem 1 (1 point per part) No proofs needed, but you should have a proof in mind!)
- (14 points) §2.1, Problems 2, 3, 27(a), 27(b), 27(c), 27(d), 42(a) (2 points each)
- (2 points) §2.2, Problem 2(a)

Additional Problems

- (2 points) Let V be a vector space, and let $B := \{v_1, \ldots, v_n\}$ be a basis. Let $W \subset V$ be a subspace. Must there exist a subset $S \subset B$ such that S is a basis for W? (If yes, prove the result. If no, give a counterexample.)

Solutions

In my solutions, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

In general, solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

1. (§1.6, 1) True or false:

   (a) The zero vector space has no basis.

   False. The zero vector space $\{0\}$ has the empty set as a basis. By definition, the empty set is linearly independent. By convention, we say that an empty sum is 0, and hence 0 is a linear combination of things in the empty set. Thus the empty set is a linearly independent set which spans the zero vector space.

   (b) Every vector space that is generated by a finite set has a basis.

   True. This is theorem 1.9 in the book.

   (c) Every vector space has a finite basis.

   False. An infinite dimensional vector space does not have a finite basis (or else it would be finite-dimensional). For example, the set of all polynomials over a field is infinite-dimensional: one basis is $\{1, x, x^2, x^3, \ldots\}$.

   (d) A vector space cannot have more than one basis.

   False. Almost every vector space has more than one basis. For example, the vector space $\mathbb{R}$ already has infinitely many bases. Any nonzero number in $\mathbb{R}$ is a basis for $\mathbb{R}$.

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.

True. This is Corollary 1 in the book (p47).

(f) The dimension of $P_n(F)$ is n.

False. One basis for $P_n(F)$ is $\{1, x, x^2, x^3, \dots, x^n\}$. By the definition of dimension, we have $\dim P_n(F) = n + 1$.

(g) The dimension of $M_{m\times n}(F)$ is $m + n$.

False. A basis for $M_{m\times n}(F)$ is given by the matrices E_{ij} for $1 \le i \le m$ and $1 \le j \le n$, where E_{ij} is the matrix with a 1 in position ij, and zeros everywhere else. There are m choices for i and n choices for j, so this basis consists of mn vectors.

(h) Suppose that V is a finite dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

True. This follows from the replacement theorem (Theorem 1.10 in the book).

(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

False. For example, take $s_1 = (1, 0), s_2 = (0, 1), s_3 = (1, 1)$, and set $S = \{s_1, s_2, s_3\} \subset \mathbb{R}^2$. Then S generates $\mathbb{R}^2$ but $(1, 1) = s_1 + s_2$, but it is also equal to $(1, 1) = s_3$.

(j) Every subspace of a finite dimensional space is finite dimensional.

True. This follows from Theorem 1.11 in the book.

(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspsace with dimension n.

True. The only 0-dimension space is the 0 vector space $\{0\}$. If $W \subset V$ is an n-dimensional subspace, then $W = V$ by Theorem 1.11. So the unique 0 and n-dimensional subspaces of V are $\{0\}$ and V respectively.

(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

True. If $|S| = n$ and S is linearly independent, then by Corollary 2(b) in the book (p48), S is a basis and hence spans V. Conversely, if $|S| = n$ and S spans V, then by Corollary 2(a), S is a basis and hence is linearly independent.

2. (§2.1, 2) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Prove that T is a linear transformation, find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems to determine whether T is one-to-one or onto.

Solution. To prove T is linear, you need to check that for any $c, a_1, a_2, a_3, a_1', a_2', a_3' \in \mathbb{R}$:

$$T(a_1 + a_1', a_2 + a_2', a_3 + a_3') = T(a_1, a_2, a_3) + T(a_1', a_2', a_3')$$

and that

$$T(ca_1, ca_2, ca_3) = cT(a_1, a_2, a_3)$$

This follows from a straightforward application of the definition of T. Details omitted.

The image of T is the subspace consisting of all $(x, y) \in \mathbb{R}^2$ satisfying $x = a_1 - a_2, y = 2a_3$. Solving this system of equations shows that it has solutions for all $x, y \in \mathbb{R}$. In other words, T is surjective, since for example $T(x, 0, \frac{y}{2}) = (x, y)$. Thus $R(T) = \mathbb{R}^2$, a basis for which is $\{e_1, e_2\} = \{(1, 0), (0, 1)\}$. Thus $\operatorname{rank}(T) = 2$. The nullspace consists of all vectors (a_1, a_2, a_3) satisfying $a_1 - a_2 = 0, 2a_3 = 0$. The general solution is: $(a_1, a_1, 0)$ (for any $a_1 \in \mathbb{R}$). Thus $N(T)$ is the 1-dimensional subspace $\{(a_1, a_1, 0) \mid a_1 \in \mathbb{R}\}$. One basis for $N(T)$ is $(1, 1, 0)$.

Thus we have $\operatorname{rank}(T) + \operatorname{nullity}(T) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$, so T verifies the dimension theorem (as it should, or else we must have made a mistake somewhere).

Finally, we saw already that T is surjective. It is clearly not one-to-one since its kernel is nonzero.

3. (§2.1, 3) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$. Prove that T is a linear transformation, find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems to determine whether T is one-to-one or onto.

Solution. We omit the proof that T is linear.

By Theorem 2.2 (in the book), the image of T is the span of $T(e_1), T(e_2)$. Using the definition of T, we can compute these to be:

$$\begin{aligned} T(e_1) &= T(1,0) = (1,0,2) \\ T(e_2) &= T(0,1) = (1,0,-1) \end{aligned}$$

These two vectors are easily seen to be linearly independent, since neither is a linear combination of the other. Thus $(1,0,2), (1,0,-1)$ is a basis for $R(T)$, and hence $R(T)$ is 2-dimensional (so T has rank 2)

The kernel of T consists of all $(a_1, a_2) \in \mathbb{R}^2$ satisfying $a_1 + a_2 = 0$ and $2a_1 - a_2 = 0$. Solving this system shows that the unique solution is $(0,0)$, so the kernel is the zero subspace, with basis the empty set, so T has nullity 0.

We have $\text{rank}(T) + \text{nullity}(T) = 2 + 0 = 2 = \dim(\mathbb{R}^2)$, so T satisfies the dimension theorem again (as it should, or else we must have made a mistake somewhere). Since $\text{Ker}(T) = 0$, T is one-to-one by Theorem 2.4. Since the image is 2-dimensional but $\mathbb{R}^3$ is 3-dimensional, T is not onto.

4. (§2.1, 27) Let V be a vector space and W_1, W_2 be subspaces such that $V = W_1 \oplus W_2$ (i.e., $V = W_1 + W_2$ and $W_1 \cap W_2 = 0$). The function $T : V \to V$ defined by $T(x) = x_1$, where $x = x_1 + x_2$ with $x_i \in W_i$, is called the *projection of* V *on* W_1.

Remark. The definition of T is unambiguous because the assumption that $W_1 \cap W_2 = 0$ implies that the decomposition $x = x_1 + x_2$ is *unique*. That is to say, x_1, x_2 are the *unique* elements in W_1, W_2 respectively satisfying $x = x_1 + x_2$. Thus, x_1, x_2 are determined by x, so it makes sense to define $T(x) = x_1$. This was proven in Homework 2 (§1.3, Problem 30). This uniqueness is crucial for this problem to make sense.

(a) Prove that T is linear and $W_1 = \{x \in V : T(x) = x\}$.

Proof. Suppose $x, x' \in V$ with $x = x_1 + x_2$ and $x' = x_1' + x_2'$ (where $x_i, x_i' \in W_i$). Then $x + x' = (x_1 + x_1') + (x_2 + x_2')$ where the first term lies in W_1 and the second lies in W_2. Thus $T(x + x') = x_1 + x_1' = T(x) + T(x')$, as desired. Similarly, if $a \in F$, then $ax = ax_1 + ax_2$ where the first term lies in W_1 and the second in W_2. Thus $T(ax) = ax_1 = aT(x)$. This shows T is linear.

If $x \in W_1$, then its decomposition $x = x_1 + x_2$ is simply $x = x + 0$ (so $x_1 = x, x_2 = 0$), and we have $T(x) = x$. This shows that $W_1 \subset \{x \in V : T(x) = x\}$. Conversely, if $x \in V$ satisfies $T(x) = x$, then since the image of T is a subset of W_1, it follows that $x \in W_1$, so $\{x \in V : T(x) = x\} \subset W_1$. Since we've proven containments in both directions, we must have equality: $W_1 = \{x \in V : T(x) = x\}$. □

(b) With notation as above, prove that $W_1 = R(T)$ and $W_2 = N(T)$.

Proof. It follows from the definition of T that $R(T) \subset W_1$. Conversely, if $x \in W_1$, then from part (a) we know that $T(x) = x$, and hence $x \in R(T)$. Thus $W_1 \subset R(T)$. Since we've proven both containments, we must have equality: $W_1 = R(T)$.

If $x \in W_2$, then the decomposition $x = x_1 + x_2$ is just $x = 0 + x$, so $x_1 = 0$, so $T(x) = x_1 = 0$. This shows that $W_2 \subset N(T)$. Conversely, if $x \in N(T)$, then $T(x) = x_1 = 0$, but that means $x = x_1 + x_2 = 0 + x_2 = x_2 \in W_2$, so $x \in W_2$. This shows $N(T) \subset W_2$. Again we have proven both containments, so we must have $W_2 = N(T)$. □

(c) Describe T if $W_1 = V$.

Solution. If $W_1 = V$, then since $W_2 \subset V$, we must have $W_1 \cap W_2 = W_2$. Since $W_1 \cap W_2 = 0$, it follows that $W_2 = 0$. Thus for any $x \in V$, we must have $x_2 = 0$, Thus $x = x_1 + x_2 = x_1 + 0 = x_1$, so $T(x) = x_1 = x$. Thus T is the identity $V \to V$.

(d) Describe T if W_1 is the zero subspace.

Solution. If $W_1 = 0$, then since $W_1 + W_2 = V$, we must have $W_2 = V$. By (b), we know that $N(T) = W_2 = V$, so T must be the zero map $V \to V$ (i.e., T sends everything to 0).

5. (§2.1, 42(a)) Let V be a vector space and W a subspace of V. Define the map $\eta : V \to V/W$ by $\eta(v) = v+W$ for $v \in V$. Prove that η is a linear transformation from V onto V/W and that $N(\eta) = W$.

Proof. Let $v_1, v_2 \in V$. Then

$$\eta(v_1 + v_2) = (v_1 + v_2) + W = (v_1 + W) + (v_2 + W) = \eta(v_1) + \eta(v_2)$$

Here, the second equality follows from the definition of addition for cosets. The first and third equalities follows from the definition of η. Similarly, for any $a \in F$, $v \in V$ we have

$$\eta(av) = av + W = a(v + W)$$

where the first equality is the definition of η, the second is the definition of scalar multiplication of cosets. This shows that η is linear.

Next we show that it is *onto* V/W. This is easy, since every element of V/W is a coset of the form $v + W$, and by definition η sends v to $v + W$. Thus η surjects onto V/W.

Finally we show that $N(\eta) = W$. If $w \in W$, then by §1.3 Exercise 31(b) in Homework 2, we have $\eta(w) = w + W = 0 + W$, which is the zero vector in the coset space V/W. Thus $W \subset N(\eta)$. In the other direction, if $\eta(v) = 0 + W$, then by definition of η, this means $v + W = 0 + W$. By §1.3 Exercise 31(b) again, we conclude that $v - 0 = v \in W$. Thus $N(\eta) \subset W$. Since we have proven both containments, we must have $N(\eta) = W$.

□

6. (§2.2, 2(a)) Let β, γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$, compute $[T]_\beta^\gamma$.

Solution. Since the domain and codomain are 2 and 3-dimensional respectively, we may write $\beta = (b_1, b_2)$ and $\gamma = (g_1, g_2, g_3)$. The matrix $[T]_\beta^\gamma$ will have two columns and three rows. The first column consists of the coefficients of $T(b_1)$ relative to the basis γ, and the second column consists of the coefficients of $T(b_2)$ relative to the basis γ. Since β, γ are both the standard bases, the first column the coefficients of $T(e_1)$ relative to the standard basis of $\mathbb{R}^3$. Since

$$T(e_1) = T(1, 0) = (2, 3, 1) = 2e_1 + 3e_2 + e_3$$

the first column has entries $2, 3, 1$. Similarly, since

$$T(e_2) = T(0, 1) = (-1, 4, 0) = -e_1 + 4e_2 + 0e_3$$

the second column has entries $-1, 4, 0$. Thus the matrix is

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

Remark. Of course in your solution you can skip the $\beta = (b_1, b_2), \gamma = (g_1, g_2, g_3)$ part, since they are both the standard bases. I included it to remind you how to compute matrices with respect to general bases.

7. (Additional Problem) Let V be a vector space, and let $B := \{v_1, \ldots, v_n\}$ be a basis. Let $W \subset V$ be a subspace. Must there exist a subset $S \subset B$ such that S is a basis for W? (If yes, prove the result. If no, give a counterexample.)

Solution. The answer is no. For example, take the standard basis $B = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$, and take $W = \text{Span}\{(1, 1)\}$. Then no subset of B gives a basis for W.

Remark. The subspaces of $\mathbb{R}^2$ can be described as follows. There's always the 0 subspace (this is the unique 0-dimensional subspace). There's also the whole space (this is the unique top-dimensional subspace, in this case 2-dimensional). Finally, there are infinitely many 1-dimensional subspaces, each one is given by a line through the origin. If B is *any* basis for $\mathbb{R}^2$, then a subset $S \subset B$ will be the basis of a 1-dimensional subspace if and only if $|S| = 1$. However since $|B| = 2$, there are only 2 subsets of size 1 of B, so only two 1-dimensional subspaces have a basis which is a subset of B. The remaining infinitely many subspaces do not have a basis which is a subset of B.