# MATH 350 Linear Algebra
# Homework 2 Solutions

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In the following, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

Each problem is worth 2 points (30 points total).

**These solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.**

§1.3, 23(a) Let $W_1, W_2 \subset V$ be subspaces. Prove that $W_1 + W_2$ is a subspace of $V$ that contains both $W_1$ and $W_2$.

*Proof.* If $w, w' \in W_1 + W_2$, we may write $w = w_1 + w_2, w' = w_1' + w_2'$ for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$. Then $w+w' = (w_1+w_1')+(w_2+w_2')$. Since each of $W_1, W_2$ are subspaces, $w_1+w_1' \in W_1$ and $w_2+w_2' \in W_2$. Thus $w + w' \in W_1 + W_2$. Similarly, for $a \in F$, $aw = aw_1 + aw_2$. Since $W_1, W_2$ are subspaces $aw_i \in W_i$ for $i = 1, 2$, so $aw \in W_1 + W_2$. Finally, for $i = 1, 2$, any $w_i \in W_i$ can be written as $0 + w_i = w_i + 0 \in W_1 + W_2$, so $W_1 + W_2$ contains both $W_1, W_2$. □

§1.3, 23(b) Prove that any subspace of $V$ that contains both $W_1, W_2$ must also contain $W_1 + W_2$.

*Proof.* Let $W \subset V$ be a subspace containing both $W_1, W_2$. This means for any $w_1 \in W_1, w_2 \in W_2$, $w_1, w_2 \in W$. Since $W$ is a subspace, it contains $w_1 + w_2$. Since $w_i$ is arbitrary in $W_i$, this shows that $W$ contains $W_1 + W_2$. □

§1.3, 30 Let $W_1, W_2$ be subspaces of $V$. Prove that $V$ is the direct sum of $W_1, W_2$ if and only if each vector in $V$ can be uniquely written as $x_1 + x_2$ where $x_i \in W_i$.

*Proof.* By definition, $V$ is the direct sum of $W_1, W_2$ if $V = W_1 + W_2$ and $W_1 \cap W_2 = 0$. (See p22 in the book for the definition).

Now suppose $V$ is the direct sum. Since $V = W_1 + W_2$, any $v \in V$ can be written $v = w_1 + w_2$ where $w_i \in W_i$. If $w_1' \in W_1, w_2' \in W_2$ satisfy $v = w_1' + w_2'$, then we would have

$$w_1 + w_2 = v = w_1' + w_2'$$

Equivalently,

$$w_1 - w_1' = w_2' - w_2$$

Since the left hand side lies in $W_1$ and the right hand side lies in $W_2$, this means $w_1 - w_1' \in W_1 \cap W_2$, so $w_1 - w_1' = 0 = w_2' - w_2$. Thus $w_i = w_i'$ for $i = 1, 2$. This shows that the decomposition $v = w_1 + w_2$ is unique.

For the converse, suppose $V$ is not the direct sum. We must show that the statement $\mathbf{P}$ := "every $v \in V$ can be written uniquely as $w_1 + w_2$ for $w_i \in W_i$" fails. Since $V$ is not the direct sum, either $V \neq W_1 + W_2$, or $W_1 \cap W_2 \neq 0$. If $V \neq W_1 + W_2$, then there is some $v \in V$, $v \notin W_1 + W_2$, which is to say that $v$ cannot be written as $w_1 + w_2$ with $w_i \in W_i$. Thus the statement **P** fails. If $W_1 \cap W_2 \neq 0$, then let $w \in W_1 \cap W_2$ be a nonzero vector. Then $w \in V$ can be written as $w = w + 0 = 0 + w$. This gives you two different ways of representing $w$ as $w_1 + w_2$ with $w_i \in W_i$, so again **P** fails. Thus we've shown that if $V$ is not the direct sum, then the statement **P** fails. □

§1.3, 31(a) Let $W$ be a subspace of a vector space $V$ over a field $F$. For any $v \in V$, let set $v + W := \{v + w : w \in W\}$ is called the *coset* of $W$ containing $v$. Prove that $v + W$ is a subspace of $V$ if and only if $v \in W$.

*Proof.* If $v + W$ is a subspace, then $0 \in v + W$, so $0 = v + w$ for some $w \in W$, so $v = -w$. Since subspaces contain inverses, this implies $v \in W$. Conversely, if $v \in W$, then we will show that $v + W = W$, and hence is a subspace. Clearly $v + W \subset W$. Moreover, for any $w \in W$, we have $w = v + (w - v)$. Since $v \in W$, $w - v \in W$, so we've shown that any $w \in W$ lies in $v + W$. This shows that $W \subset v + W$, so $W = v + W$, hence $v + W$ is a subspace.

□

§1.3, 31(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

*Proof.* Suppose $v_1 - v_2 \in W$, then $v_2 - v_1 = -(v_1 - v_2) \in W$. We want to show that $v_1 + W \subset v_2 + W$ and that $v_2 + W \subset v_1 + W$. Since $v_1 - v_2 \in W$, for any $w \in W$ we have $v_1 + w = v_2 + (v_1 - v_2 + w) \in v_2 + W$. This shows the first containment. Since $v_2 - v_1 \in W$, for any $w \in W$ we have $v_2 + w = v_1 + (v_2 - v_1 + w) \in v_1 + W$, which shows the second containment.

Conversely, suppose $v_1 + W = v_2 + W$. Then $v_1 \in v_1 + W = v_2 + W$, so we can write $v_1 = v_2 + w$, which is to say that $v_1 - v_2 = w \in W$. □

§1.3, 31(c) Addition and scalar multiplication by elements of $F$ can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of $W$ as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$, and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$. Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W) \qquad \text{and} \qquad a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

*Proof.* By the definition of addition, and scalar multiplication, we must show that $(v_1 + v_2) + W = (v_1' + v_2') + W$, and $av_1 + W = av_1' + W$.

Since $v_i + W = v_i' + W$, by 31(b) we have $v_i - v_i' \in W$ (for $i = 1, 2$). But this means that $(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$. Again using 31(b), we find that $(v_1 + v_2) + W = (v_1' + v_2') + W$. Similarly, we know that $a(v_1 - v_1') = av_1 - av_1' \in W$, so by 31(b) we have $av_1 + W = av_1' + W$, as desired.

□

§1.3, 31(d) Prove that the set $S$ is a vector space with the operations defined in (c). This vector space is called the *quotient space V mod W* and is denoted by $V/W$.

*Proof.* (omitted) □

§1.5, 2(c) Determine if the set $\{x^3+2x^2, -x^2+3x+1, x^3-x^2+2x-1\} \subset P_3(\mathbb{R})$ is linearly independent. **Solution.** We are interested in understanding the set of solutions $(a_1, a_2, a_3)$ of the equation

$$a_1(x^3+2x^2)+a_2(-x^2+3x+1)+a_3(x^3-x^2+2x-1)=0$$

Collecting like terms, we are looking at the linear system

$$\begin{aligned} a_1+a_3 &= 0 \\ 2a_1-a_2-a_3 &= 0 \\ 3a_2+2a_3 &= 0 \\ a_2-a_3 &= 0 \end{aligned}$$

By definition of linear independence, the set is linearly independent if and only if the *only* solution to the system is $a_1=a_2=a_3=0$.

The general solution can be obtained using the techniques of Math 250, but in this case one can find it without too much work. The last equation forces $a_2=a_3$, so we may replace all instances of $a_3$ with $a_2$. The first equation forces $a_1+a_2=0$, so we may replace all instances of $a_2$ with $-a_1$. The second equation then becomes $2a_1+a_1+a_1=0$, so $a_1=-a_2=-a_3=0$. Thus the coefficients are all forced to be 0, so the set is linearly independent.

§1.5, 8 Let $S=\{(1,1,0),(1,0,1),(0,1,1)\}$ be a subset of the vector space $F^3$.

(a) Prove that if $F=\mathbb{R}$, then $S$ is linearly independent.

*Proof.* With the technology of §1.6, we could show this by showing that $S$ spans $\mathbb{R}^3$, but we don't have that at our disposal, so we must proceed more directly. Again, we wish to understand the solutions $(a_1,a_2,a_3)$ to the equation

$$a_1(1,1,0)+a_2(1,0,1)+a_3(0,1,1)=(0,0,0)$$

This turns into the linear system

$$\begin{aligned} a_1+a_2 &= 0 \\ a_1+a_3 &= 0 \\ a_2+a_3 &= 0 \end{aligned}$$

Again this can be solved using the techniques of Math 250. Here we proceed more organically: The first equation forces $a_2=-a_1$, the second forces $a_3=-a_1$, so $a_3=a_2=-a_1$. Using this, the last equation forces $a_3+a_3=2a_3=0$, so $a_3=\frac{1}{2}\cdot 2a_3=\frac{1}{2}\cdot 0=0$. This makes crucial use of the fact that we can divide by 2. □

(b) Prove that if $F$ has characteristic 2, then $S$ is linearly dependent.

*Proof.* Recall that $F$ has characteristic 2 if $1+1=0$. Note that the crucial difference with the case $F=\mathbb{R}$ is that in the final step of the part 9a), $a_3+a_3=0$ no longer forces $a_3=0$. This is bceause over a field of characteristic 2, for any $a\in F$, $a+a=2a=0$. The idea is to use this observation to come up with a linear dependence relation for $S$. In this case, the linear dependence relation is simple: take $a_1=a_2=a_3=1$. Then we have

$$(1,1,0)+(1,0,1)+(0,1,1)=(2,2,2)=(0,0,0)\in F^3$$

Here the coefficients are all 1. Note that in any field, $0\neq 1$. □

§1.5, 10 Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

**Solution.** There are many such examples. The first observation is that since $\dim\mathbb{R}^3=3$, for three vectors in $\mathbb{R}^3$ to be linearly dependent, they cannot span $\mathbb{R}^3$ (this uses the technology of §1.6). Since none of them is a multiple of another, none of them can be 0, and they also cannot span a 1-dimensional space. Thus we must look for 3 vectors which span a 2-dimensional subspace of $\mathbb{R}^3$. Here is an example:

$$(1,0,0),(0,1,0),(1,1,0)$$

They span the 2-dimensional subspace $\{(x,y,0) : x,y \in \mathbb{R}\} \subset \mathbb{R}^3$, and clearly none of them is a multiple of another.

§1.5, 18 Let $S$ be a set of nonzero polynomials in $P(F)$ such that no two have the same degree. Prove that $S$ is linearly independent.

*Proof.* Suppose for some $a_1,\dots,a_n \in F$ and $f_1,\dots,f_n \in S$, we have

$$a_1 f_1 + \cdots + a_n f_n = 0 \tag{1}$$

Suppose the $f_i$'s are numbered so that $\deg f_1 < \deg f_2 < \cdots < \deg f_n$. Let $S_r$ be the set $\{f_1,\dots,f_r\}$. We will show by induction that each $S_r$ is linearly independent. The base case $r = 1$ holds because $f_1 \neq 0$, so $S_1 = \{f_1\}$ is independent. Suppose $S_r$ is linearly independent (for $r < n$), then since $\deg f_{r+1} > \deg f_r$, $f_{r+1} \notin \operatorname{Span} S_r$, so by Theorem 1.7 in the book, $S_{r+1}$ is linearly independent. By induction, this shows that each $S_r$ is linearly independent. In particular, $S_n$ is. □

§1.6, 11 Let $u, v$ be distinct vectors of a vector space $V$. Show that if $\{u, v\}$ is a basis for $V$ and $a, b$ are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for $V$.

*Proof.* Note that since $\{u, v\}$ is a basis for $V$, we have $\dim V = 2$.

First we show $S := \{u + v, au\}$ is a basis. Since $a \neq 0$, $\frac{1}{a} \cdot au = u \in \operatorname{Span} S$, so $-u \in \operatorname{Span} S$, so $v = (u + v) + (-u) \in \operatorname{Span} S$, so $\{u, v\} \subset \operatorname{Span} S$. Since $u, v$ spans $V$, $\operatorname{Span} S = V$. Now we must show that $S$ is linearly independent. We could do this directly, which would be annoying, or we can appeal to our newly acquired technology. Specifically, Corollary 2(a) in §1.6 of the book implies that because $S$ is a spanning set which has size equal to $\dim V = 2$, it must be a basis.

Next we show that $S' := \{au, bv\}$ is a basis. The same argument as above shows that $u, v \in \operatorname{Span} S'$, so $S'$ is again a spanning set. Since it has size equal to $\dim V = 2$, Corollary 2(a) implies that it is a basis.

□

§1.6, 32(a) Find examples of subspaces $W_1, W_2$ of $\mathbb{R}^3$ such that $\dim W_1 > \dim W_2 > 0$ and $\dim(W_1 \cap W_2) = \dim W_2$.

**Solution.** You can take any two nonzero subspaces satisfying $W_1 \supsetneq W_2$. In this case $W_1 \cap W_2 = W_2$, so $\dim(W_1 \cap W_2) = \dim(W_2)$. For example, take $W_1 = \mathbb{R}^3$ and $W_2$ any nonzero subspace (e.g., $\{(x,y,0) : x,y \in \mathbb{R}\}$).

§1.6, 32(b) Find examples of subspaces $W_1, W_2$ of $\mathbb{R}^3$ such that $\dim W_1 > \dim W_2 > 0$ and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$.

**Solution.** It turns out this happens if and only if $W_1 \cap W_2 = 0$ (Exercise: prove this!). For example, you can take

$$W_1 = \{(x,y,0) : x,y \in \mathbb{R}\} \qquad W_2 = \{(0,0,z) : z \in \mathbb{R}\}$$

§1.6, 32(c) Find examples of subspaces $W_1, W_2$ of $\mathbb{R}^3$ such that $\dim W_1 > \dim W_2 > 0$ and $\dim(W_1 + W_2) < \dim W_1 + \dim W_2$.

**Solution.** Here we want to take subspaces with nontrivial intersection. For example, we can take

$$W_1 = \{(x,y,0) : x,y \in \mathbb{R}\} \qquad W_2 = \{(x,0,0) : x \in \mathbb{R}\}$$

Then $W_1 + W_2 = W_1$, so $\dim(W_1 + W_2) = \dim W_1 = 2$, but $\dim W_1 + \dim W_2 = 2 + 1 = 3$.