MATH 350 Linear Algebra
Homework 1 Solutions

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September 16, 2022

In the following, any solution which says anything to the effect of "details omitted" is not a complete solution – in your writeup you would be expected to fill in the details. Otherwise, you can treat the solution as an example of something that would earn you full credit. However some solutions include more detail than is necessary.

Each problem is worth 2 points (30 points total).

These solutions may have typos/errors. I guarantee that by the end of the semester there will be at least one typo. Please let me know ASAP if you find any, so I can correct it.

1. (§1.2, Problem 2)

   The zero vector is just the 3×4 matrix with all zeroes.

2. (§1.2, Problem 4(a))

   The sum is just
$$\begin{bmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{bmatrix}$$

3. (§1.2, Problem 4(f))

   The sum is obtained by collecting like terms:
$$-x^3 + 7x^2 + 4$$

4. (§1.2, Problem 8) In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.

   Proof. We will apply distributivity three times:
$$\begin{aligned} (a+b)(x+y) &= (a+b)x + (a+b)y \\ &= ax + bx + (a+b)y \\ &= ax + bx + ay + by \end{aligned}$$
   $\square$

5. (§1.2 Problem 9) See `goo.gl/WFWgzX`

6. (§1.2 Problem 12) A real-valued function f defined on the real line $\mathbb{R}$ is even if $f(-t) = f(t)$ for all $t \in \mathbb{R}$. Prove that the set of even functions $\mathbb{R} \to \mathbb{R}$ with the operations of addition and scalar multiplication is a vector space.

   Proof. This is not a complete solution.

   The first part involves showing that addition and scalar multiplication makes sense on even functions. You have to show that if f, g are even, then so is $f + g$ and af for any $a \in \mathbb{R}$. Next, you have to check that these operations satisfy the axioms of a vector space.

VS1 For any two even functions f, g, $f + g = g + f$. This means we have to check that for all $x \in \mathbb{R}$, $(f+g)(x) = (g+f)(x)$. From the definition of addition of functions, this is the same as checking that $f(x) + g(x) = g(x) + f(x)$, but this follows from the commutativity of addition of real numbers.

VS2 Similar to VS1 (omitted).

VS3 The element 0 is the zero function $\underline{0}$, defined by $\underline{0}(x) = 0$ for all $x \in \mathbb{R}$. (Check this satisfies the desired property)

VS4 The additive inverse $-f$ is just the function $(-f)(x) = -f(x)$. (Check this satisfies the desired property)

VS5 For any function f, $(1 \cdot f)(x) = f(x)$, so $1 \cdot f = f$.

VS6 For any $a, b \in F$ and even function f, $(ab)f$ is the function defined by $((ab)f)(x) = (ab)f(x) = a \cdot b \cdot f(x)$ (this is the product of three real numbers). Similarly, bf is the function defined by $(bf)(x) = b \cdot f(x)$, and $a(bf)$ is the function defined by $(a(bf))(x) = a \cdot (bf)(x) = a \cdot b \cdot f(x)$. Thus $(ab)f = a(bf)$.

VS7 Similar to VS6 (omitted)

VS8 Similar to VS6 (omitted)

□

7. (§1.2 Problem 17) Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$ where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (a_1, 0)$$

Is V a vector space over F with these operations?

Proof. The answer is no. There are many ways to justify this. Perhaps the easiest is that taking $c = 1 \in F$, then VS5 requires that $1(a_1, a_2) = (a_1, a_2)$ by VS5, in particular VS5 requires $1(1,1) = (1,1)$, but the definition of the action would have $1(1,1) = (1,0)$. In a field, $1 \neq 0$.

Here, you should make sure that your answer makes sense for any field F (not just for $F = \mathbb{R}$). □

8. (§1.3, Problem 5) Prove that $A + A^t$ is symmetric for any square matrix A.

Proof. If the i,jth coordinate of A is A_{ij}, then the i,jth coordinate of A^t is A_{ji}. Thus the i,jth coordinate of $A + A^t$ is $A_{ij} + A_{ji}$, and the i,jth coordinate of $(A + A^t)^t$ is $A_{ji} + A_{ij}$. By commutativity of addition, this shows that

$$A + A^t = (A + A^t)^t$$

Alternatively, you could just say that $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$. □

9. (§1.3, Problem 8(a)) Determine if $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$ is a subspace of $\mathbb{R}^3$.

   This is a subspace. Need to check the criteria for a subset to be a subspace (details omitted).

10. (§1.3, Problem 8(b)) Determine if $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$ is a subspace of $\mathbb{R}^3$.

    This is *not* a subspace. The easiest way to see this is to note that $(0, 0, 0) \notin W_2$.

11. (§1.4, Problem 1(a)) True or false: The zero vector is a linear combination of any nonempty set of vectors.

    True. For any nonempty set of vectors S, pick $s \in S$, then $\vec{0} = 0 \cdot s$ is a linear combination of vectors in S.

12. (§1.4, Problem 1(c)) True or false: If S is a subset of a vector space V, then $\mathrm{Span}(S)$ equals the intersection of all subspaces of V that contain S.

    True. For this problem you don't need to prove this, but here is a proof:

Proof. To show that two sets are equal, we need to show that each one is contained in the other. From the Quiz, we know that if $W \subset V$ is a subspace containing S, then W contains $\mathrm{Span}(S)$. Thus $\mathrm{Span}(S)$ is contained in every subspace containing S, and so $\mathrm{Span}(S)$ is contained in the intersection of all of them. It remains to show that the intersection is contained in $\mathrm{Span}(S)$. Since $\mathrm{Span}(S)$ is itself a subspace containing S, we find that $\mathrm{Span}(S)$ is one of the subspaces being intersected. Thus the intersection must be contained in $\mathrm{Span}(S)$. □

13. (§1.4, Problem 1(f))

    False. The system $0 = 1$ is a system of linear equation (with 1 equation), which does not have a solution. Or, you can take the system $x = 0, x = 1$. Or $x + y = 0, x + y = 1$. What is true is that any *homogeneous* system of linear equations has a solution (for a homogeneous system, setting every variable equal to 0 always gives a solution).

14. (§1.4, Problem 4(a)) Determine if $f(x) := x^3 - 3x + 5$ is a linear combination of $g_1(x) := x^3 + 2x^2 - x + 1$ and $g_2(x) := x^3 + 3x^2 - 1$.

    We want to find a_1, a_2 so that $f = a_1 g_1(x) + a_2 g_2$. Extracting the coefficients of x^n of the two sides, we want to solve the system:

$$\begin{aligned} 1 &= a_1 + a_2 \\ 0 &= 2a_1 + 3a_2 \\ -3 &= -a_1 \\ 5 &= a_1 - a_2 \end{aligned}$$

    This can be solved using the techniques of Math 250. Alternatively, the third equation forces $a_1 = 3$, and combined with the first we have $a_2 = -2$. One can check that this also satisfies the remaining two equations, so f is indeed a linear combination of g_1, g_2.

15. (§1.4, Problem 5(g)) Determine if the vector $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ is in the span of $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

    Let S_1, S_2, S_3 be the matrices in S (given in order). We want to find constants a_1, a_2, a_3 such that $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = a_1 S_1 + a_2 S_2 + a_3 S_3$. Again, extracting the coefficients of the matrices, we obtain the system:

$$\begin{aligned} 1 &= a_1 + a_3 \\ 2 &= a_2 + a_3 \\ -3 &= -a_1 \\ 4 &= a_2 \end{aligned}$$

    The last two equations forces $a_1 = 3$, $a_2 = 4$. Combined with the first, we get $a_3 = -2$. This satisfies the second equation, so this is a solution. The general solution can be found using the methods of Math 250 (though this question does not ask for a general solution).