MATH 350 Name: ******************

FINAL EXAM SOLUTIONS

This is an closed book, closed notes exam. No calculators are allowed.

Useful shorthand: Feel free to write:

- "LI" instead of "linearly independent"
- "LD" instead of "linearly dependent"
- "v.s." instead of "vector space"
- "f.d.", or "fin. dim." instead of "finite dimensional". You can also write "$\dim V < \infty$" for "V is finite dimensional".
- "evalue, vector, espace" for "eigenvalue, eigenvector, eigenspace".

If you use this, make sure you write **very clearly**.

Reminders: If T is a linear operator on a vector space V, and $W \subset V$ is T-invariant, then T_W denotes the linear operator $T_W : W \to W$ given by $T_W(w) = T(w)$. Also, if $v \in V$, then the T-invariant subspace generated by v is $\langle v \rangle_T := \text{Span}\{v, Tv, T^2v, \ldots\}$. Recall that if $x \in V$ is a generalized λ-eigenvector, then the cycle generated by x is the set $C_x := \{(T - \lambda I)^{p-1}x, \ldots, (T - \lambda I)x, x\}$, where p is the smallest positive integer such that $(T - \lambda I)^p x = 0$. The vector $(T - \lambda I)^{p-1}x$ is called the *initial vector* of the cycle C_x, and x is called the *end vector*.

For a linear operator $T : V \to V$, if β is a basis of V, then $[T]_\beta$ denotes the matrix of T w.r.t. the basis β. If $V = \mathbb{R}^n$, std $:= \{e_1, \ldots, e_n\}$ denotes the standard basis.

If you are asked to prove and if and only if ("$\iff$"), then you must prove *both directions*. If you are asked to prove that two sets A, B are equal, then you must prove $A \subset B$ and $B \subset A$.

Every vector space is implicitly over some field F. Recall the definition of a field:

Definition 0.0.1 (Fields)**.** A field F is a set with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$, such that the following hold for all $a, b, c \in F$:

(F1) $a + b = b + a$ and $a \cdot b = b \cdot a$

(F2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(F3) There exist distinct elements "0" and "1" in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

(F4) For each $a \in F$ and *nonzero* $b \in F$, there exist elements $c, d \in F$ such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

(F5) $a \cdot (b + c) = a \cdot b + a \cdot c$

In F4, c is called the negative of a, denoted "$-a$", and d is called the multiplicative inverse of b, denoted "b^{-1}" or "$1/b$".

1. (12 points, 1 point each) Label the following statements (T)rue or (F)alse. No justification required.

   (a) Suppose $\{x, y, w, z\} \subset \mathbb{R}^4$ spans $\mathbb{R}^4$, then $\{x, y, z, w\}$ is linearly independent.

   Solution. TRUE.

   (b) Suppose V is a vector space and $\{x_1, x_2, x_3\} \subset V$ a subset such that $\{x_i, x_j\}$ is linearly independent whenever $i \neq j$ ($i, j \in \{1, 2, 3\}$). Then $\{x_1, x_2, x_3\}$ must be linearly independent.

   Solution. FALSE. E.g., take $(1, 0), (0, 1), (1, 1)$ in $\mathbb{R}^2$

   (c) If $T : V \to W$ is linear and $S \subset V$ is linearly independent, then $T(S)$ is also linearly independent.

   Solution. FALSE. This is only true if T is 1-1.

   (d) If $T : V \to W$ is linear and $S \subset V$ spans V, then $T(S)$ spans W.

   Solution. FALSE. This is only true if T is onto.

   (e) $\{0\}$ is a basis for the zero vector space.

   Solution. FALSE. 0 is never a member of any basis.

   (f) If $T : V \to V$ is a linear operator, then for any two polynomials $f(t), g(t)$, $f(T)g(T) = g(T)f(T)$.

   Solution. TRUE.

   (g) If $T : V \to V$ is a linear operator with $\chi_T(t) = (\lambda_1 - t)^{m_{\lambda_1}}(\lambda_2 - t)^{m_{\lambda_2}} \cdots (\lambda_k - t)^{m_{\lambda_k}}$ (where $\lambda_i \in F$ for each i), then $\det(T) = \lambda_1^{m_{\lambda_1}} \lambda_2^{m_{\lambda_2}} \cdots \lambda_k^{m_{\lambda_k}}$.

   Solution. TRUE.

   (h) If T is a linear operator on a finite dimensional vector space V and $W \subset V$ is a T-invariant subspace, then $\chi_{T_W}(t)$ divides $\chi_T(t)$.

   Solution. TRUE.

   (i) Let T be a linear operator on a finite dimensional vector space V with split characteristic polynomial. Let $\lambda_1, \ldots, \lambda_k$ be its distinct roots, then $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$.

   Solution. TRUE.

   (j) Let T be a linear operator on a finite dimensional vector space V. Then T is similar to an upper triangular matrix if and only if χ_T is split.

   Solution. TRUE.

   (k) If $T : V \to V$ is linear, $\dim V < \infty$, and β a basis for V, then $N(T)$ is isomorphic to $N([T]_\beta)$.

   Solution. TRUE.

   (l) Let T be a linear operator on a vector space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. If S_i is a linearly independent subset of K_{λ_i}, then $S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.

   Solution. TRUE.

2. Let A, B be two linear operators on a finite dimensional vector space V. For the following, if the answer is yes, prove it. If no, give a counterexample.

   (a) (4 pts) If AB is an isomorphism, must A and B both be isomorphisms?

   Solution. Yes. Suppose B is not an isomorphism. Then there is a nonzero $v \in N(B)$, so that $Bv = 0$, so $ABv = A0 = 0$, so $N(AB)$ is also nonzero, so AB is not an isomorphism. If A is not an isomorphism, then it is not onto, so there is a vector $v \in V$ which is not in the range (or image) of A. But then v is also not in the range of AB, so in this case AB is also not an isomorphism.

   (b) (4 pts) If $AB = 0$, must at least one of A, B be zero?

   Solution. No. For example take $A = B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, then $AB = A^2 = 0$ but neither A nor B are zero.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T(x, y, z) = (-x - 3y - z, -19y - 2z, 9y + z)$$

   (a) (4 pts) Find the matrix $[T]_{\text{std}}$.

   Solution. The matrix is

$$\begin{bmatrix} -1 & -3 & -1 \\ 0 & -19 & -2 \\ 0 & 9 & 1 \end{bmatrix}$$

   (b) (4 pts) Find the characteristic polynomial $\chi_T(t)$ and the determinant $\det(T)$.

   Solution. $\chi_T(t) = (-1 - t)((-19 - t)(1 - t) - (-18)) = (t + 1)(t^2 + 18t - 19 + 18) = -(t + 1)(t^2 + 18t - 1) = -t^3 + 19t^2 + 17t + 1$. The determinant is just $\det(T) = \chi_T(0) = 1$.

   (c) (3 pts) Is T diagonalizable? Justify your answer.

   Solution. Yes. By the quadratic formula, the polynomial $t^2 + 18t - 1$ has two distinct roots, neither of which are -1, so $\chi_T(t)$ has 3 distinct roots. Thus T is diagonalizable.

   (d) (4 pts) Is T invertible? If it is, use the Cayley-Hamilton theorem to express T^{-1} as a linear combination of I, T, T^2. If not, explain why not.

   Solution. Yes. Since $\det(T)$ is nonzero, T is invertible. By the Cayley-Hamilton theorem, we have $-T^3 + 19T^2 + 17T + I = 0$, so $-T^2 + 19T + 17I + T^{-1} = 0$, so $T^{-1} = T^2 - 19T - 17I$.

4. (5 pts) Find an example of a matrix $A \in M_4(\mathbb{R})$ whose minimal polynomial equals its characteristic polynomial.

   Solution. You can take any matrix with split characteristic polynomial such that the dot diagram associated to each eigenvalue consists of a single column. E.g., you can take

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   This matrix A has a single eigenvalue, A is its own Jordan canonical form, with Jordan basis the standard basis, which is also a single cycle. Its minimal polynomial is t^4 which is equal to its characteristic polynomial.

5. A linear operator N which satisfies $N^k = 0$ for some $k \geq 1$ is called *nilpotent*.

   (a) (4 pts) If N is a nilpotent operator on a vector space V, prove that 0 is the only eigenvalue of N (i.e., show that 0 is an eigenvalue and N has no eigenvalues other than 0).

   Solution. If $N^k = 0$, then the null space of N must be nonzero. Indeed, take any nonzero vector $v \in V$, and suppose $p \geq 1$ is the smallest positive integer such that $N^p v = 0$. We know such a p exists since $N^k v = 0$ (so $p \leq k$). Then $NN^p v = N^{p+1} v = 0$, so $N^p v$ is a nonzero vector in the null space of N. Thus 0 is an eigenvalue.

   Now suppose v is a λ-eigenvector for some scalar λ. Then $Nv = \lambda v$, so $N^k v = \lambda^k v$, but $N^k v = 0$, so $\lambda^k v = 0$. Since v is an eigenvector, $v \neq 0$, so this implies that $\lambda^k = 0$, so $\lambda = 0$.

   (b) (4 pts) Prove that if N is a nilpotent operator on $\mathbb{R}^n$, then $\chi_N(t) = t^n$. Hint: Be careful, why can't $\chi_N(t)$ be something like $t(t^2+1)$?

   Solution. Let $N_{\mathbb{C}}$ be the operator on $\mathbb{C}^n$ given by the same matrix as N. Then $\chi_{N_{\mathbb{C}}}(t) = \chi_N(t)$ splits as a polynomial over $\mathbb{C}$. By (a), the only root of $\chi_{N_{\mathbb{C}}}(t)$ is 0, so $\chi_{N_{\mathbb{C}}}(t) = \chi_N(t) = t^n$.

   (c) (4 pts) Prove that if $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator such that 0 is the only eigenvalue, then T is nilpotent.

   Solution. If the only eigenvalue of T is 0, the only root of $\chi_T(t)$ is 0. Since $\chi_T(t)$ splits over $\mathbb{C}$, this means $\chi_T(t) = t^n$. By Cayley-Hamilton, $\chi_T(T) = 0$, which is exactly to say that $T^n = 0$.

   (d) (4 pts) Give an example of a matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ for which 0 is the only eigenvalue, but A is not nilpotent. Justify why your example is not nilpotent.

   Solution. The hint to part (b) is the solution here. One can take A to be the operator on $\mathbb{R}^3$ given by

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Its characteristic polynomial is $-t(t^2+1)$, so by (b) it is not nilpotent.

   Another way to see that it is not nilpotent is to note that $\mathrm{Span}\{e_1, e_2\}$ is an invariant subspace on which A is invertible.

6. (7 pts) Find the determinant of the matrix

$$A = \det \begin{bmatrix} -4 & 9 & -14 & 15 \\ 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \end{bmatrix}$$

Solution. Typo here: The "det" should be deleted. One should evaluate this determinant using the method of elementary row operations. Swapping the first and second rows, we then use the resulting first row to kill 0 below it (in the first column). We then continue with this process for the remaining columns. The status after doing this to each column is:

$$\begin{bmatrix} 1 & -2 & 3 & -12 \\ 0 & 1 & -2 & -33 \\ 0 & 2 & 1 & -41 \\ 0 & 4 & 7 & -77 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 3 & -12 \\ 0 & 1 & -2 & -33 \\ 0 & 0 & 5 & 25 \\ 0 & 0 & 15 & 55 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 3 & -12 \\ 0 & 1 & -2 & -33 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 15 & 55 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 3 & -12 \\ 0 & 1 & -2 & -33 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{bmatrix}$$

Along the way, the only operations that change the determinant is swapping two rows, and dividing a row by 5. Together, this modifiies the determinant by dividing by -5. Since the determinant of the last matrix above is -20, $\det(A) = (-5 \cdot -20) = 100$.

7. Suppose a matrix $A \in M_3(\mathbb{R})$ satisfies $A^3 = I$.

   (a) (3 pts) Show that $\det(A) = 1$.

   Solution. Since determinants are multiplicative, $\det(A^3) = \det(AAA) = \det(A)\det(A)\det(A) = \det(A)^3$. Then

   $$\det(A)^3 = \det(A^3) = \det(I) = 1$$

   Thus $\det(A)$ is a real number whose cube is 1, so it must be 1.

   (b) (6 pts) Prove that 1 must be an eigenvalue of A. Hint: What are the possiblities for the minimal polynomial of A as a matrix over $\mathbb{C}$? Note that the polynomial $t^3 - 1$ factors as $t^3 - 1 = (t-1)(t^2+t+1)$, and $t^2+t+1 = (t-\zeta)(t-\zeta^2)$, where $\zeta \in \mathbb{C} - \mathbb{R}$ and satisfies $\zeta^3 = 1$.

   Solution. Let $f(t) := t^3 - 1$, and let $p(t)$ denote the minimal polynomial of A as a matrix over $\mathbb{C}$. Then A satisfies $f(t)$, so $p(t)$ divides $f(t)$. Over $\mathbb{C}$, $f(t)$ factors as $(t-1)(t^2+t+1) = (t-1)(t-\zeta)(t-\zeta^2)$. If $t-1$ divides $p(t)$, then since $p(t)$ divides $\chi_A(t)$, 1 is a root of $\chi_A(t)$, so 1 is an eigenvalue. If $t-1$ doesn't divide $p(t)$, then the only way for this to happen is if $p(t) = (t-\zeta), (t-\zeta^2)$, or $(t-\zeta)(t-\zeta^2)$. Since A has coefficients in $\mathbb{R}$ and ζ, ζ^2 are not in $\mathbb{R}$, A cannot possibly satisfy $t-\zeta$ or $t-\zeta^2$. If $p(t) = (t-\zeta)(t-\zeta^2)$, then its constant term is 1. Writing $\chi_A(t) = -p(t)(t-a)$, $\chi_A(t)$ has constant term a. Since the constant term of $\chi_A(t)$ is also $\det(A) = 1$, it follows that $a = 1$, so $(t-1)$ divides $\chi_A(t)$, so 1 is an eigenvalue.

8. Let $U = \text{Span}\{(1,1,0),(1,0,1))\} \subset \mathbb{R}^3$, and let $W = \text{Span}\{(1,0,0),(4,1,2)\}$.

   (a) (5 pts) Find a basis for $U \cap W$. Hint: Set up a vector equation that describes elements of $U \cap W$.

   Solution. An element $v \in U \cap W$ can be simultaneously written as a linear combination of $(1,1,0)$ and $(1,0,1)$, as well as a linear combination of $(1,0,0),(4,1,2)$. Thus we can write

$$v = a(1,1,0) + b(1,0,1) = c(1,0,0) + d(4,1,2)$$

   The latter equation is equivalent to $a(1,0,0) + b(1,0,1) - c(1,0,0) - d(4,1,2)$. Equivalently, this is

$$\begin{bmatrix} 1 & 1 & -1 & -4 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Solving this, we find that there is exactly one free variable d, and a, b, c are subject to the conditions $a = d, b = 2d, c = a + b - 4d$. Thus, it follows that the intersection is 1-dimensional, spanned by any nonzero vector. Such a nonzero vector can be obtained by setting $d = 1$, so that $a = 1, b = 2, c = -1$, corresponding to the vector

$$1 \cdot (1,1,0) + 2 \cdot (1,0,1) = (3,1,2) = -1 \cdot (1,0,0) + 1 \cdot (4,1,2)$$

   Thus $\{(3,1,2)\}$ is a basis for $U \cap W$.

   (b) (3 pts) Find a matrix A such that $U \cap W = N(A)$.

   Solution. This question is unintentionally a bit silly, but at least it is easy given part (a). You can take the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Without using part (a), one can also consider the matrix associated to the map f coming from 9c, but this requires some massaging.

   (c) (5 pts) Show that $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

   Solution. We know that $\dim U = \dim W = 2$, and $\dim(U \cap W) = 1$. The latter can be deduced from (a), but it can also be deduced from the fact that the intersection of two planes in $\mathbb{R}^3$ must have dimension at least 1 and at most 2. The intersection is 2 dimensional if and only if the planes are equal, but since $(1,0,0)$ is not in U, the planes are not equal, so the intersection has dimension 1. Indeed, one can easily check that $(1,1,0),(1,0,1),(1,0,0)$ spans $\mathbb{R}^3$, so $(1,0,0) \notin U$. This also shows that $U + W = \mathbb{R}^3$, which proves the desired equality.

9. Let U, W be finite dimensional vector spaces. The direct product $U \times W$ is the vector space $U \times W := \{(u,w) \mid u \in U, w \in W\}$ where addition and scalar multiplication are given by

$$(u,w) + (u',w') := (u + u', w + w'), \quad a \cdot (u,w) := (au, aw) \quad \text{for any } a \in F,\ u,u' \in U,\ w,w' \in W$$

   (a) (2 pts) Prove that the zero vector in $U \times W$ is $(0,0)$.

   Solution. If $(u,w) \in U \times W$, so $(u,w) + (0,0) = (u+0, w+0) = (u,w)$. This shows that $(0,0)$ is the zero vector.

(b) (5 pts) (Continued from previous page) Prove that $\dim U \times W = \dim U + \dim W$

Solution. Let $\beta = \{\beta_1, \dots, \beta_n\}$ and $\gamma = \{\gamma_1, \dots, \gamma_m\}$ be bases of U, W respectively. Then it's clear that $\alpha := \{(\beta_1, 0), \dots, (\beta_n, 0), (0, \gamma_1), \dots, (0, \gamma_n)\}$ spans $U \times W$. It remains to show that α is linearly independent. A general linear combination takes the form

$$(b_1\beta_1 + \cdots + b_n\beta_n, c_1\gamma_1 + \cdots + c_n\gamma_n)$$

But this is equal to the zero vector $(0,0)$ if and only if each coordinate is 0, which forces all the coefficients b_i, c_i to be 0 since β, γ are linearly independent. Thus it follows that α is linearly independent.

(c) (5 pts) Assume furthermore that U, W are both subspaces of a finite dimensional space V. Let f be the map

$$f : U \times W \longrightarrow U + W \quad \text{defined by} \quad f(u, w) = u - w$$

Prove that f is linear and surjective.

Solution. Let $v \in U + W$, then we can write $v = u + w$ for $u \in U, w \in W$. Then $f(u, -w) = u + w = v$, so f is surjective. To check that it is linear, let $u' \in U, w' \in W$, and $a \in F$, then

$$f(a(u, w)) = f(au, aw) = au - aw = a(u - w) = af(u, w)$$

and

$$f((u, w) + (u', w')) = f(u + u', w + w') = u + u' - w - w' = u - w + u' - w' = f(u, w) + f(u', w')$$

This shows that f is linear.

(d) (5 pts) With the same assumptions as in part (c), Prove that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Hint: Show that the null space of f is isomorphic to $W_1 \cap W_2$.

Solution. The null space of f consists of precisely the vectors (u, w) such that $u - w = 0$, i.e. $u = w$, so the null space of f is exactly the set $N(f) = \{(v, v) \mid v \in U \cap W\}$. The map $U \cap W \to N(f)$ given by $v \mapsto (v, v)$ is linear, and obviously both 1-1 and onto. Thus $U \cap W$ is isomorphic to $N(f)$, and hence $\dim(U \cap W) = \dim N(f)$. Using the rank-nullity theorem,

$$\dim U + \dim W = \dim(U \times W) = \dim N(f) + \dim R(f) = \dim(U \cap W) + \dim(U + W)$$

Rearranging a bit gets us the desired formula.

10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear map with matrix

$$[T]_{\text{std}}^{\text{std}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Let T^* be the linear map $T^* : \mathcal{L}(\mathbb{R}^2, \mathbb{R}) \to \mathcal{L}(\mathbb{R}^3, \mathbb{R})$ given by

$$T^*(f) = f \circ T$$

The vector space $\mathcal{L}(\mathbb{R}^k, \mathbb{R})$ is sometimes called the *dual space of* $\mathbb{R}^k$, and is sometimes denoted $(\mathbb{R}^k)^*$. Let $e_1^*, \ldots, e_k^* \in \mathcal{L}(\mathbb{R}^k, \mathbb{R})$ be defined by

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(a) (7 pts) Show that $e_1^*, \ldots, e_k^*$ is a basis for $\mathcal{L}(\mathbb{R}^k, \mathbb{R})$. Hint: Recall that the map $\Psi : \mathcal{L}(\mathbb{R}^k, \mathbb{R}) \longrightarrow M_{1\times k}(\mathbb{R})$ sending $g : \mathbb{R}^k \to \mathbb{R}$ to the matrix $[g]_{\text{std}}^{\text{std}}$ is an *isomorphism*. What is $\Psi(e_i^*)$?

Solution. The matrix of e_i^* is just the row vector (or $1 \times k$ matrix) $e_i^T \in M_{1\times k}(\mathbb{R})$, where T denotes transpose (if we view e_i as a column vector). It's obvious that the row vectors $\{e_1^T, \ldots, e_k^T\}$ form a basis for $M_{1\times k}(\mathbb{R})$, so the image of $\{e_1^*, \ldots, e_k^*\}$ is a basis. Since Ψ is an isomorphism, this implies that $e_1^*, \ldots, e_k^*$ are a basis for $\mathcal{L}(\mathbb{R}^k, \mathbb{R})$.

(b) (8 pts) Let $\beta = \{e_1^*, e_2^*\}$ and $\gamma = \{e_1^*, e_2^*, e_3^*\}$. Find the matrix $[T^*]_\beta^\gamma$.

Solution. One checks that

$$T^*(e_1^*) = [1\ 0] \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [1\ 2\ 3] = 1e_1^* + 2e_2^* + 3e_3^*$$

and

$$T^*(e_2^*) = [0\ 1] \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [4\ 5\ 6] = 4e_1^* + 5e_2^* + 6e_3^*$$

Thus

$$[T^*]_\beta^\gamma = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

This is just the transpose of $[T]$.

11. (15 pts) Find a Jordan canonical form, a Jordan canonical basis, the minimal polynomial, characteristic polynomial, and determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & 3 & -12 \\ -1 & 0 & -1 \end{bmatrix}$$

Hint: 1 is an eigenvalue.

Solution. Cofactor expanding along the bottom row, the characteristic polynomial is

$$\chi_A(t) = -1(-12)+(-1-t)((2-t)(3-t)+4) = 12-(t+1)(t^2-5t+6+4) = 12-(t^3-4t^2+5t+10) = -t^3+4t^2-5t+2$$

Since 1 is an eigenvalue, $(t-1)$ divides $\chi_A(t)$. Dividing by $(t-1)$, we find that $\chi_A(t) = -(t-1)^2(t-2)$.

One computes that $N(A-2I) = \text{Span}\{(3, 0, -1\}$, so $(3, 0, -1)$ is a basis of cycles for K_2.

Next, one finds that $N(A-I) = \text{Span}\{(2, -2, -1)\}$, so the dot diagram for the eigenvalue 1 consists of a single column. Next, we find that $N((A-I)^2) = \text{Span}\{(1, 1, 0), (0, 4, 1)\}$. It follows that we can take the cycle basis for K_1 to be the cycle generated by $(1, 1, 0)$. The corresponding cycle is $\{(2, -2, -1), (1, 1, 0)\}$. Thus a Jordan canonical basis is $\{(2, -2, -1), (1, 1, 0), (3, 0, -1)\}$. The associated Jordan canonical form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

With determinant 2, and minimal polynomial $(t-1)^2(t-2)$.

12. Suppose $T : V \to V$ is a linear operator with $\chi_T(t) = t^2(t-1)^4(t-2)^6$. Let $U_\lambda := T - \lambda I$. Suppose that $\dim N(U_0) = 1$, $\dim N(U_1) = 3$, $\dim N(U_2) = 3$, and $\dim N(U_2^2) = 5$.

(a) (4 pts) Find the dot diagrams for the eigenvalues $0, 1, 2$. Is T diagonalizable?

Solution. The dot diagrams for 0,1,2 are, respectively,

$$\lambda = 0 : \begin{array}{c} \bullet \\ \bullet \end{array} \qquad \lambda = 1 : \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \end{array} \qquad \lambda = 2 : \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

(b) (4 pts) Find a Jordan canonical form of T.

Solution. From the dot diagram, a Jordan canonical form is

$$\begin{bmatrix} 0 & 1 & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & \\ & & 1 & 1 & & & & & & & & \\ & & 0 & 1 & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & 2 & 1 & 0 & & & \\ & & & & & & 0 & 2 & 1 & & & \\ & & & & & & 0 & 0 & 2 & & & \\ & & & & & & & & & 2 & 1 & \\ & & & & & & & & & 0 & 2 & \\ & & & & & & & & & & & 2 \end{bmatrix}$$

Where all unlabelled entries are 0.

(c) (5 pts) Let $p(t)$ be the minimal polynomial of T. Since $p(t)$ divides $\chi_T(t)$, there exist integers $r_0, r_1, r_2 \geq 0$ such that $p(t) = t^{r_0}(t-1)^{r_1}(t-2)^{r_2}$, where $r_0 \leq 2, r_1 \leq 4, r_2 \leq 6$. Find r_0, r_1, r_2, and briefly justify why the resulting polynomial is the polynomial of minimum positive degree such that $p(T) = 0$.

Solution. $r_0 = 2, r_1 = 2, r_2 = 3$. Since each U_λ acts as an isomorphism on K_μ for $\mu \neq \lambda$, the r_λ's can be individually computed for each K_λ. We need $r_0 = 2$ since U_0 does not kill K_0 but U_0^2 does. We need $r_1 = 2$ since U_1 doesn't kill K_1 but U_1^2 does. We need $r_2 = 3$ since U_2, U_2^2 do not kill K_2, but U_2^3 does.

(d) (5 pts) Suppose there is a vector $v \in N(U_2^3) - N(U_2^2)$. Suppose $w \in N(U_2^2)$ is a vector which does not lie in $\operatorname{Span} C_v + N(U_2)$. Show that $C_v \cap C_w = \emptyset$ and $C_v \cup C_w$ is linearly independent. Is $C_v \cup C_w$ a basis of K_2? If not, briefly describe how to extend $C_v \cup C_w$ to a basis for K_2.

Solution. Recall that $C_v = \{U_2^2 v, U_2 v, v\}$. Clearly $w \notin C_v$. Suppose $U_2 w$ lies in the span of $U_2^2 v$, then $U_2 w = aU_2^2 v$ for some scalar a. Then $U_2(w - aU_2 v) = 0$, so $w - aU_2 v \in N(U_2)$, so

$$w = aU_2 v + (w - aU_2 v)$$

But $aU_2 v \in \operatorname{Span} C_v$, and $w - aU_2 v \in N(U_2)$, so that $w \in \operatorname{Span} C_v + N(U_2)$, which contradicts our assumption on w. It follows that $U_2 w$ is independent of $U_2^2 v$, so that the cycles C_v, C_w are linearly independent and disjoint by theorem 7.6 in the book. Since K_2 is 6-dimensional and $C_v \cup C_w$ only has 5 elements, it does not give a basis of K_2. To complete it to a basis, you can add any vector $z \in N(U_2) - \text{Span}\{U_2^2 v, U_2 w\}$ to $C_v \cup C_w$.