MATH 350 Name: ____________________

EXAM 2

This is an closed book, closed notes exam. No calculators are allowed.

Useful shorthand: Feel free to write:

- "LI" instead of "linearly independent"
- "LD" instead of "linearly dependent"
- "LT" instead of "linear transformation"
- "v.s." instead of "vector space"
- "f.d.", or "fin. dim." instead of "finite dimensional". You can also write "$\dim V < \infty$" for "V is finite dimensional".

If you use this, make sure you write **very clearly**.

Reminders: A *linear operator* on a vector space V is a linear map $T : V \to V$. If T is a linear operator on a vector space V, and $W \subset V$ is T-invariant, then T_W denotes the linear operator $T_W : W \to W$ given by $T_W(w) = T(w)$. Also, if $v \in V$, then the T-invariant subspace generated by v is $\langle v \rangle_T := \mathrm{Span}\{v, Tv, T^2v, \dots\}$.

For a linear operator $T : V \to V$, if β is a basis of V, then $[T]_\beta$ denotes the matrix of T w.r.t. the basis β. If $V = \mathbb{R}^n$, std $:= \{e_1, \dots, e_n\}$ denotes the standard basis.

If you are asked to prove and if and only if ("$\iff$"), then you must prove *both directions*. If you are asked to prove that two sets A, B are equal, then you must prove $A \subset B$ and $B \subset A$.

Every vector space is implicitly over some field F. Recall the definition of a field:

Definition 0.0.1 (Fields)**.** A field F is a set with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$, such that the following hold for all $a, b, c \in F$:

(F1) $a + b = b + a$ and $a \cdot b = b \cdot a$

(F2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(F3) There exist distinct elements "0" and "1" in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

(F4) For each $a \in F$ and *nonzero* $b \in F$, there exist elements $c, d \in F$ such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

(F5) $a \cdot (b + c) = a \cdot b + a \cdot c$

In F4, c is called the negative of a, denoted "$-a$", and d is called the multiplicative inverse of b, denoted "b^{-1}" or "$1/b$".

1. (24 points, 3 points each) Label the following statements (T)rue or (F)alse. Include a short justification of your answer.

   (a) If λ is an eigenvalue of a linear operator T on V, then $E_\lambda := \{v \in V \mid Tv = \lambda v\}$ is the span of the λ-eigenvectors.

   (b) If T is a linear operator on a 2-dimensional vector space, then T is diagonalizable if and only if it has at least one eigenvalue.

   (c) If $\dim V < \infty$, $T : V \to V$ is linear, and β, β' are two bases for V, then $[T]_\beta$ and $[T]_{\beta'}$ have the same characteristic polynomial.

   (d) If T is a linear operator on a finite dimensional vector space, then T is 1-1 if and only if it is onto.

   (e) Let T be a linear operator on a vector space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. If S_i is a linearly independent subset of E_{λ_i}, then $S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.

   (f) If $A \in M_n(F)$ and $\mu \in F$, then $\det(\mu A) = \mu \det(A)$.

   (g) If $A, B \in M_n(F)$, then $\det(AB) = \det(BA)$.

   (h) The linear map $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(f(x)) = f'(x)$ has no eigenvalues.

2. (10 pts) Let $T : V \to V$ be a linear operator. Let μ, λ be two distinct eigenvalues of T. Show that $E_\mu \cap E_\lambda = 0$.

3. (a) (8 pts) Show that 0 is an eigenvalue of the matrix X with a 1-dimensional eigenspace. Hint: Don't try to compute the characteristic polynomial.

$$X = \begin{bmatrix} 1 & -3 & -1 & 2 \\ 3 & -8 & -1 & 5 \\ 5 & -14 & 0 & 10 \\ -2 & 6 & 5 & -3 \end{bmatrix}$$

(b) (3 pts) What is $\det(X)$?

4. (10 pts) Is the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

diagonalizable? If so, find a diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by:

$$T(x, y, z, w) = (x + y + 2z - w, x + 2y + 3z - w, -x + 3y + 2z + w, 3y + 2z + 2w)$$

(a) (5 pts) Let $W := \langle e_1 \rangle_T$ be the T-invariant subspace generated by e_1. Show that $\dim W = 3$. Hint: It may help to write down the matrix $[T]_{\text{std}}$.

(b) (4 pts) Find the matrix of T_W with respect to the basis $\{e_1, Te_1, T^2e_1\}$ of W.

(c) (4 pts) Find the characteristic polynomial of T_W.

(d) (5 pts) Is the characteristic polynomial of T split (over $\mathbb{R}$)? Why?

(e) (5 pts) Is T_W diagonalizable? Is T diagonalizable? Why?

6. (10 points) Suppose T is a linear operator on a 4-dimensional vector space V with $\det(T) = 0$. Suppose $W \subset V$ is a 3-dimensional T-invariant subspace such that T_W is diagonalizable and $\det(T_W) \neq 0$. Prove that T is diagonalizable.

7. Every complex number can be represented uniquely as a sum $a + bi$, where $a, b \in \mathbb{R}$. One can check that $\mathbb{C}$ is a vector space over $\mathbb{R}$ with basis $\beta = \{1, i\}$. If $z = a + bi \in \mathbb{C}$, write $m_z : \mathbb{C} \to \mathbb{C}$ for the "multiplication map" $m_z(w) = zw$.

   (a) (2 pts) Show that $m_z : \mathbb{C} \to \mathbb{C}$ is a linear map of $\mathbb{R}$-vector spaces.

   (b) (4 pts) Find the matrix $[m_z]_\beta$ in terms of a and b.

   (c) (3 pts) Compute the eigenvalues of $[m_z]_\beta$ viewed as a matrix in $M_2(\mathbb{C})$.

   (d) (3 pts) Find the matrix $[m_i]_\beta$, where $i \in \mathbb{C}$ is the imaginary unit. Describe how m_i acts geometrically on the complex plane.