MATH 350 Name: *******************

EXAM 2 SOLUTIONS

This is an closed book, closed notes exam. No calculators are allowed.

Useful shorthand: Feel free to write:

- "LI" instead of "linearly independent"
- "LD" instead of "linearly dependent"
- "LT" instead of "linear transformation"
- "v.s." instead of "vector space"
- "f.d.", or "fin. dim." instead of "finite dimensional". You can also write "$\dim V < \infty$" for "V is finite dimensional".

If you use this, make sure you write **very clearly**.

Reminders: A *linear operator* on a vector space V is a linear map $T : V \to V$. If T is a linear operator on a vector space V, and $W \subset V$ is T-invariant, then T_W denotes the linear operator $T_W : W \to W$ given by $T_W(w) = T(w)$. Also, if $v \in V$, then the T-invariant subspace generated by v is $\langle v \rangle_T := \text{Span}\{v, Tv, T^2v, \ldots\}$.

For a linear operator $T : V \to V$, if β is a basis of V, then $[T]_\beta$ denotes the matrix of T w.r.t. the basis β. If $V = \mathbb{R}^n$, std $:= \{e_1, \ldots, e_n\}$ denotes the standard basis.

If you are asked to prove and if and only if ("$\iff$"), then you must prove *both directions*. If you are asked to prove that two sets A, B are equal, then you must prove $A \subset B$ and $B \subset A$.

Every vector space is implicitly over some field F. Recall the definition of a field:

Definition 0.0.1 (Fields)**.** A field F is a set with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$, such that the following hold for all $a, b, c \in F$:

(F1) $a + b = b + a$ and $a \cdot b = b \cdot a$

(F2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(F3) There exist distinct elements "0" and "1" in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

(F4) For each $a \in F$ and *nonzero* $b \in F$, there exist elements $c, d \in F$ such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

(F5) $a \cdot (b + c) = a \cdot b + a \cdot c$

In F4, c is called the negative of a, denoted "$-a$", and d is called the multiplicative inverse of b, denoted "b^{-1}" or "$1/b$".

1. (24 points, 3 points each) Label the following statements (T)rue or (F)alse. Include a short justification of your answer.

(a) If λ is an eigenvalue of a linear operator T on V, then $E_\lambda := \{v \in V \mid Tv = \lambda v\}$ is the span of the λ-eigenvectors.

Solution. TRUE. Every nonzero vector in E_λ is a λ-eigenvector, and any vector space is spanned by its nonzero vectors.

(b) If T is a linear operator on a 2-dimensional vector space, then T is diagonalizable if and only if it has at least one eigenvalue.

Solution. FALSE. Take for example $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ on $\mathbb{R}^2$.

(c) If $\dim V < \infty$, $T : V \to V$ is linear, and β, β' are two bases for V, then $[T]_\beta$ and $[T]_{\beta'}$ have the same characteristic polynomial.

Solution. TRUE. Let $Q := [I]_\beta^{\beta'}$, then $[T]_\beta = Q^{-1}[T]_{\beta'}Q$, so $[T]_\beta$ is similar to $[T]_{\beta'}$, so they have the same characteristic polynomial.

(d) If T is a linear operator on a finite dimensional vector space, then T is 1-1 if and only if it is onto.

Solution. TRUE. By rank-nullity, $\text{nullity}(T) + \text{rank}(T) = \dim V$. Thus T is 1-1 if and only if $\text{nullity}(T) = 0$ if and only if $\text{rank}(T) = \dim V$ if and only if T is onto.

(e) Let T be a linear operator on a vector space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. If S_i is a linearly independent subset of E_{λ_i}, then $S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.

Solution. TRUE. This is theorem 5.5 in the book.

(f) If $A \in M_n(F)$ and $\mu \in F$, then $\det(\mu A) = \mu \det(A)$.

Solution. FALSE. Take $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $\det(2A) = 4$ which is not equal to $2\det(A) = 2$.

(g) If $A, B \in M_n(F)$, then $\det(AB) = \det(BA)$.

Solution. TRUE. We know $\det(AB) = \det(A)\det(B)$ and $\det(BA) = \det(B)\det(A)$, but $\det(A)\det(B) = \det(B)\det(A)$ since determinants lie in F and $xy = yx$ in any field.

(h) The linear map $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(f(x)) = f'(x)$ has no eigenvalues.

Solution. FALSE, it has 0 has a eigenvalue, since $T(x) = 0$.

2. (10 pts) Let $T : V \to V$ be a linear operator. Let μ, λ be two distinct eigenvalues of T. Show that $E_\mu \cap E_\lambda = 0$.

Proof. Suppose $v \in E_\mu \cap E_\lambda$. Since $v \in E_\mu$, we know $Tv = \mu v$. Since $v \in E_\lambda$, we know $Tv = \lambda v$. Thus we have $\mu v = \lambda v$, so $\mu v - \lambda v = \vec{0}$, so

$$(\mu - \lambda)v = \vec{0}$$

Since $\mu \neq \lambda$, $\mu - \lambda \neq 0$, so it has an inverse in F. Multiplicying both sides of the above equation by its inverse, we get

$$v = (\mu - \lambda)^{-1}\vec{0} = \vec{0}$$

This shows that any vector in $E_\mu \cap E_\lambda$ must be equal to 0, as desired. □

3. (a) (8 pts) Show that 0 is an eigenvalue of the matrix X with a 1-dimensional eigenspace. Hint: Don't try to compute the characteristic polynomial.

$$X = \begin{bmatrix} 1 & -3 & -1 & 2 \\ 3 & -8 & -1 & 5 \\ 5 & -14 & 0 & 10 \\ -2 & 6 & 5 & -3 \end{bmatrix}$$

Proof. Computing determinants of 4x4 matrices without a lot of zeroes is very cumbersome. Instead, we will directly calculate the null space $N(X)$ and show that it is 1-dimensional. For this we simply row reduce. Using the first row to make zeroes in the first column, we get

$$\rightsquigarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Doing the same with the second and third columns, we get

$$\rightsquigarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that there are 3 pivots, so X has rank 3 and hence nullity 1. Since the eigenspace for the eigenvalue 0 is $N(X - 0I) = N(X)$, this shows that the 0-eigenspace is 1-dimensional. □

(b) (3 pts) What is $\det(X)$?

Solution. Since X has positive nullity, it is not invertible, so $\det(X) = 0$. In general, we have:

$$\det(X) = 0 \iff 0 \text{ is an eigenvalue of } X \iff N(X) \neq 0$$

4. (10 pts) Is the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

diagonalizable? If so, find a diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$.

Solution. Since A has two identical rows, clearly $\text{rank}(A) \leq 2$, so $\text{nullity}(A) \geq 1$, so 0 is an eigenvalue of A. However to check for diagonalizability, we need to understand its other eigenvalues. Its characteristic polynomial can be computed by cofactor expansion along the first row:

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 0 & 2 \\ -1 & 3-t & 1 \\ 1 & 0 & 2-t \end{bmatrix} = (1-t)\det \begin{bmatrix} 3-t & 1 \\ 0 & 2-t \end{bmatrix} + 2\det \begin{bmatrix} -1 & 3-t \\ 1 & 0 \end{bmatrix}$$

$$\cdots = (1-t)(3-t)(2-t) - 2(3-t) = (t^2 - 3t + 2)(3-t) - 2(3-t) = (t^2 - 3t)(3-t) = -t(t-3)^2$$

It follows that the eigenvalues of A are 0, 3 with multiplicities 1 and 2 respectively. To check for diagonalizability, it suffices to show that the 3-eigenspace E_3 is 2-dimensional. For this, we need to compute

$$N(A - 3I) = N\left(\begin{bmatrix} -2 & 0 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \right) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

It follows that E_3 is 2-dimensional, with basis $(1,0,1), (0,1,0)$, so A is diagonalizable. To compute Q, we must find a basis for each eigenspace. We already did this for E_3. For E_0, we have $N(A - 0I) = N(A)$. This can be computed by row-reduction. Performing a sequence of elementary row operations, we get

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \rightsquigarrow Z_1 A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow Z_2 Z_1 A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Where Z_1, Z_2 are each a product of elementary matrices. Clearly

$$N(Z_2 Z_1 A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

But since $N(A) = N(Z_2 Z_1 A)$, it follows that $(-2, -1, 1)$ is a basis for $E_0 = N(A)$. This can also be verified directly by checking that $A \cdot (-2, -1, 1) = \vec{0}$. Thus, we find that

$$\beta = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an eigenbasis for A. Thus we can take Q to be any matrix which sends the standard basis to the eigenbasis (in whatever order). If we send $e_i \mapsto \beta_i$, then we should take Q to be

$$Q = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad \text{in which case} \qquad Q^{-1}AQ = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by:

$$T(x, y, z, w) = (x + y + 2z - w, x + 2y + 3z - w, -x + 3y + 2z + w, 3y + 2z + 2w)$$

(a) (5 pts) Let $W := \langle e_1 \rangle_T$ be the T-invariant subspace generated by e_1. Show that $\dim W = 3$. Hint: It may help to write down the matrix $[T]_{\text{std}}$.

Solution. The matrix for T is:

$$[T] = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & -1 \\ -1 & 3 & 2 & 1 \\ 0 & 3 & 2 & 2 \end{bmatrix}, \text{ and } e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, Te_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, T^2 e_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, T^3 e_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

It's easy to see that the first three vectors are linearly independent, whereas $T^3 e_1 = 2T^2 e_1 - Te_1$. Thus $W = \langle e_1 \rangle_T$ is 3-dimensional.

(b) (4 pts) Find the matrix of T_W with respect to the basis $\{e_1, Te_1, T^2 e_1\}$ of W.

Solution. Let $\beta = \{e_1, Te_1, T^2 e_1\}$. By Theorem 5.21 in the book, we have

$$[T_W]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

(c) (4 pts) Find the characteristic polynomial of T_W.

Solution. By Theorem 5.21 in the book, the characteristic polynomial of T_W is $\chi_{T_W}(t) = (-1)^3(t - 2t^2 + t^3) = -t(t^2 - 2t + 1) = -t(t-1)^2$.

(d) (5 pts) Is the characteristic polynomial of T split (over $\mathbb{R}$)? Why?

Solution. Yes, because $\chi_{T_W}(t)$ must divide $\chi_T(t)$. This means that $\chi_T(t) = \chi_{T_W}(t) f(t) = -t(t-1)^2 f(t)$ for some polynomial $f(t)$. Since χ_T has degree 4 and χ_{T_W} has degree 3, $f(t)$ is degree 1, so $\chi_T(t)$ is split.

(e) (5 pts) Is T_W diagonalizable? Is T diagonalizable? Why?

Solution. The eigenvalues of T_W are 0,1. The 1-eigenspace E_1 of T_W has the same dim. as

$$N\left([T_W]_\beta - I_3\right) = N\left(\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

which clearly has dimension 1. Thus the 1-eigenspace of E_1 has dimension 1 which is less than the multiplicity of the eigenvalue 1. It follows that T_W is not diagonalizable.

We claim that this implies that T is not diagonalizable. Suppose for the sake of a contradiction that T is diagonalizable. Then V has an eigenbasis, so every $w \in W$ can be written as $a_1 v_{w,1} + \dots a_r v_{w,r}$, where the $v'_{w,i}$ are eigenvectors of T with distinct eigenvalues (if there are multiple eigenvectors for the same eigenvalue involved in the linear combination, you can treat their sum as a single eigenvector). By §5.4 Exercise 23, this implies that $v_{w,i}$ lies in W for every $w \in W$ and every i. Thus, W is spanned by eigenvectors, so a maximal linear independent subset is an eigenbasis. This shows that T_W would be diagonalizable, which we already found to be false.

6. (10 points) Suppose T is a linear operator on a 4-dimensional vector space V with $\det(T) = 0$. Suppose $W \subset V$ is a 3-dimensional T-invariant subspace such that T_W is diagonalizable and $\det(T_W) \neq 0$. Prove that T is diagonalizable.

Proof. Since $\det(T) = 0$, 0 is an eigenvalue of T, so t divides $\chi_T(t)$. Since $\det(T_W) \neq 0$, 0 is not an eigenvalue of T_W, so t does not divide $\chi_{T_W}(t)$. Since $\chi_{T_W}(t)$ has degree 3 and divides $\chi_T(t)$, this means that t can divide $\chi_T(t)$ at most once, so it must divide $\chi_T(t)$ exactly once. In fact we must have

$$\chi_T(t) = -t \cdot \chi_{T_W}(t)$$

Since T_W is diagonalizable, for each root λ of $\chi_{T_W}(t)$, the λ-eigenspace of T_W has dimension equal to the multiplicity of λ as a root of χ_{T_W}. Since 0 is not a root of χ_{T_W}, the multiplicity of λ as a root of χ_{T_W} is the same as its multiplicity as a root of χ_T. Since the λ-eigenspace for T_W is contained in the λ-eigenspace for T (and is already the maximum possible, equal to the multiplicity), it follows that every eigenspace has dimension equal to the multiplicity, so T is diagonalizable.

□

7. Every complex number can be represented uniquely as a sum $a + bi$, where $a, b \in \mathbb{R}$. One can check that $\mathbb{C}$ is a vector space over $\mathbb{R}$ with basis $\beta = \{1, i\}$. If $z = a + bi \in \mathbb{C}$, write $m_z : \mathbb{C} \to \mathbb{C}$ for the "multiplication map" $m_z(w) = zw$.

(a) (2 pts) Show that $m_z : \mathbb{C} \to \mathbb{C}$ is a linear map of $\mathbb{R}$-vector spaces.

Proof. For additivity:

$$m_z(w + w') = z(w + w') = zw + zw' = m_z(w) + m_z(w')$$

Here we used the distributivity axiom "(F5)" of fields (see page 1). For scalar-multiplicativity:

$$m_z(aw) = zaw = azw = am_z(w)$$

Here we used the commutativity axiom "(F1)". □

(b) (4 pts) Find the matrix $[m_z]_\beta$ in terms of a and b.

Solution. We know $m_z(\beta_1) = m_z(1) = z = a + bi$, and $m_z(\beta_2) = m_z(i) = iz = ai + bi^2 = -b + ai$. Thus, we have

$$[m_z]_\beta = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

(c) (3 pts) Compute the eigenvalues of $[m_z]_\beta$ viewed as a matrix in $M_2(\mathbb{C})$.

Solution. We have $\chi_{[m_z]_\beta}(t) = (a - t)^2 + b^2 = t^2 - at^2 + a^2 + b^2$. By the quadratic formula, its roots are $a + bi, a - bi$, which are the eigenvalues of $[m_z]_\beta$.

(d) (3 pts) Find the matrix $[m_i]_\beta$, where $i \in \mathbb{C}$ is the imaginary unit. Describe how m_i acts geometrically on the complex plane.

Solution. From the formula in part (b), we have $[m_i]_\beta = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. It acts on $\mathbb{C}$ by rotating everything counterclockwise around the origin by 90 degrees.