MATH 350 Name: ______________________

EXAM 1

This is an closed book, closed notes exam. No calculators are allowed.

Useful shorthand: Feel free to write:

- "LI" instead of "linearly independent"
- "LD" instead of "linearly dependent"
- "LT" instead of "linear transformation"
- "v.s." instead of "vector space"
- "f.d.", or "fin. dim." instead of "finite dimensional". You can also write "$\dim V < \infty$" for "V is finite dimensional".

If you use this, make sure you write **very clearly**.

Synonyms: Remember that *null space* is a synonym for *kernel*; *range* is a synonym for *image*; *nullity* is a synonym for the dimension of the kernel; *rank* is a synonym for the dimension of the image; *injective* is a synonym for *one-to-one*; *surjective* is a synonym for *onto*; *bijective* is a synonym for *one-to-one and onto*.

Reminders: If A, B are sets, then $A \times B := \{(a, b) \mid a \in A, b \in B\}$. We have $|A \times B| = |A| \times |B|$. If A is a set and $n \geq 1$ is an integer, then $A^n := \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$.

If you are asked to prove and if and only if ("$\iff$"), then you must prove *both directions*. If you are asked to prove that two sets A, B are equal, then you must prove $A \subset B$ and $B \subset A$.

Every vector space is implicitly over some field F. Recall the definition of a field:

Definition 0.0.1 (Fields)**.** A field F is a set with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$, such that the following hold for all $a, b, c \in F$:

(F1) $a + b = b + a$ and $a \cdot b = b \cdot a$

(F2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(F3) There exist distinct elements "0" and "1" in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

(F4) For each $a \in F$ and *nonzero* $b \in F$, there exist elements $c, d \in F$ such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

(F5) $a \cdot (b + c) = a \cdot b + a \cdot c$

In F4, c is called the negative of a, denoted "$-a$", and d is called the multiplicative inverse of b, denoted "b^{-1}" or "$1/b$".

1. (30 points, 3 points each) Label the following statements (T)rue or (F)alse. Include a short justification of your answer.

   (a) Every vector space has a unique basis.

   Solution. False. Every vector space has a basis, but the basis is rarely unique.

   (b) If $W_1, W_2 \subset V$ are two n-dimensional subspaces such that $W_1 \neq W_2$. Then $\dim(W_1 \cap W_2) < n$.

   (c) If $W_1, W_2 \subset V$ are subspaces, then $W_1 \cup W_2$ is a subspace.

   (d) If $S \subset V$ is a subset, then some subset of S is a basis for $\mathrm{Span}(S)$.

   (e) If $W \subset V$ is a subspace, then $\mathrm{Span}(W) = W$.

   (f) Let S be an infinite subset of a vector space V. Then every element of $\mathrm{Span}(S)$ is a linear combination of *finitely* many vectors in S.

   (g) Let V be a finite dimensional vector space. Let $S \subset V$ be a spanning set and $L \subset V$ be linearly independent. Then $|L| \leq |S|$.

   (h) If $f : V \to W$ is a linear transformation and V is finite-dimensional, then

   $$\dim \ker(f) + \dim \mathrm{im}(f) = \dim V$$

   (i) If $f : V \to W$ is a linear transformation and $\dim W < \dim V < \infty$, then f cannot be 1-1.

   (j) If W_1, W_2 are subspaces of a vector space V, then $W_1 \subset W_2$ if and only if $\dim W_1 \leq \dim W_2$.

2. (15 points)
   (a) (4p) Let V be a vector space, and let $v \in V$ be a nonzero vector. Prove that for any $a \in F$, $av = 0 \iff a = 0$

   (b) (4p) Let V be a vector space, and let $v \in V$ be a nonzero vector. Prove that for any $a, b \in F$, $av = bv \iff a = b$

   (c) (7p) Let $\mathbb{F}_3$ be a field with 3 elements. How many 1-dimensional subspaces are there in $\mathbb{F}_3 \times \mathbb{F}_3$? Justify your answer.

3. (20 points) Let $f : V \to W$ be a linear transformation. In the following two proofs, do not invoke any theorems about linear transformations. You may of course use the definition of a linear transformation, and the extension that $f(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f(x_i)$ for any $a_1, \dots, a_n \in F$ and $x_1, \dots, x_n \in V$.
   (a) (10p) Suppose f is one-to-one, and $S \subset V$ is a subset. Prove that S is linearly independent if and only if $f(S)$ is linearly independent. (Recall: $f(S) := \{f(s) \mid s \in S\}$)

(b) (10p) Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and that f is both one-to-one and onto. Prove that $f(\beta)$ is a basis for W.

4. (25 points) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$f(x, y, z) = (x + 2y + 3z, 5x + 5z, -x + y)$$

Let $\beta = ((1, 0, 0), (0, 1, 0), (0, 0, 1))$ be the standard ordered basis of $\mathbb{R}^3$. Let γ be the ordered basis $((1, 0, 0), (0, 1, 2), (1, 1, 1))$.

(a) (5p) Find the matrix of f relative to the standard ordered basis β. This matrix is denoted $[f]_\beta^\beta$ (or sometimes just $[f]_\beta$ for short).

(b) (2p) Compute the matrix product $[f]_\beta^\beta \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $x, y, z \in \mathbb{R}$. Your answer should be a matrix whose entries are in terms of x, y, z.

(c) (5p) Find the second column of the matrix $[f]_\beta^\gamma$.

(d) (5p) Find a basis for $\mathrm{im}(f)$. Explain why it is a basis.

(e) (5p) Find a basis for $\ker(f)$. Explain why it is a basis.

(f) (3p) Is f one-to-one? Is f onto? Why?

5. (10 points) Let V, W be nonzero finite dimensional vector spaces. Recall that $\mathcal{L}(V, W)$ denotes the space of linear transformations $V \to W$. Let $v \in V$ be a nonzero vector. Let $\mathcal{L}_v \subset \mathcal{L}(V, W)$ be the subset

$$\mathcal{L}_v := \{f \in \mathcal{L}(V, W) \mid f(v) = 0\}$$

(a) (3p) Show that $\mathcal{L}_v \subset \mathcal{L}(V, W)$ is a subspace.

(b) (7p) Show that $\dim \mathcal{L}_v = \dim \mathcal{L}(V, W) - \dim(W)$. (Hint: Consider the map $e_v : \mathcal{L}(V, W) \to W$ sending $f \mapsto f(v)$. Show that e_v is linear. What is its image? kernel?)