MATH 350 Name: *******************

EXAM 1 SOLUTIONS

This is an closed book, closed notes exam. No calculators are allowed.

Useful shorthand: Feel free to write:

- "LI" instead of "linearly independent"
- "LD" instead of "linearly dependent"
- "LT" instead of "linear transformation"
- "v.s." instead of "vector space"
- "f.d.", or "fin. dim." instead of "finite dimensional". You can also write "$\dim V < \infty$" for "V is finite dimensional".

If you use this, make sure you write **very clearly**.

Synonyms: Remember that *null space* is a synonym for *kernel*; *range* is a synonym for *image*; *nullity* is a synonym for the dimension of the kernel; *rank* is a synonym for the dimension of the image; *injective* is a synonym for *one-to-one*; *surjective* is a synonym for *onto*; *bijective* is a synonym for *one-to-one and onto*.

Reminders: If A, B are sets, then $A \times B := \{(a, b) \mid a \in A, b \in B\}$. We have $|A \times B| = |A| \times |B|$. If A is a set and $n \geq 1$ is an integer, then $A^n := \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$.

If you are asked to prove and if and only if ("$\iff$"), then you must prove *both directions*. If you are asked to prove that two sets A, B are equal, then you must prove $A \subset B$ and $B \subset A$.

Every vector space is implicitly over some field F. Recall the definition of a field:

Definition 0.0.1 (Fields)**.** A field F is a set with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$, such that the following hold for all $a, b, c \in F$:

(F1) $a + b = b + a$ and $a \cdot b = b \cdot a$

(F2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(F3) There exist distinct elements "0" and "1" in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

(F4) For each $a \in F$ and *nonzero* $b \in F$, there exist elements $c, d \in F$ such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

(F5) $a \cdot (b + c) = a \cdot b + a \cdot c$

In F4, c is called the negative of a, denoted "$-a$", and d is called the multiplicative inverse of b, denoted "b^{-1}" or "$1/b$".

1. (30 points, 3 points each) Label the following statements (T)rue or (F)alse. Include a short justification of your answer.

   (a) Every vector space has a unique basis.

   Solution. False. Every vector space has a basis, but the basis is rarely unique.

   (b) If $W_1, W_2 \subset V$ are two n-dimensional subspaces such that $W_1 \neq W_2$. Then $\dim(W_1 \cap W_2) < n$.

   Solution. True. Write $W_{12} := W_1 \cap W_2$. Clearly $W_{12} \subset W_1$, so $\dim(W_{12}) \leq n$. If $\dim(W_{12}) = n$, then W_{12} is a full dimensional subspace of W_1, so $W_{12} = W_1$. Applying the same logic, if $\dim(W_{12}) = n$, then W_{12} is a full dimensional subspace of W_2, so $W_{12} = W_2$. This means $W_1 = W_{12} = W_2$, contradicting the hypothesis that $W_1 \neq W_2$.

   (c) If $W_1, W_2 \subset V$ are subspaces, then $W_1 \cup W_2$ is a subspace.

   Solution. False. Take the union of the two axes in $\mathbb{R}^2$. Clearly it is not a subspace, since for example $e_1 + e_2$ does not lie in either axis.

   (d) If $S \subset V$ is a subset, then some subset of S is a basis for $\operatorname{Span}(S)$.

   Solution. True, this is theorem 1.9 in the book.

   (e) If $W \subset V$ is a subspace, then $\operatorname{Span}(W) = W$.

   Solution. True. Use the alternate characterization of $\operatorname{Span}(S)$ as the intersection of all subspaces containing S.

   (f) Let S be an infinite subset of a vector space V. Then every element of $\operatorname{Span}(S)$ is a linear combination of *finitely* many vectors in S.

   Solution. True. By the definition of linear combination, a single linear combination can only involve finitely many vectors.

   (g) Let V be a finite dimensional vector space. Let $S \subset V$ be a spanning set and $L \subset V$ be linearly independent. Then $|L| \leq |S|$.

   Solution. True, this is the replacement theorem.

   (h) If $f : V \to W$ is a linear transformation and V is finite-dimensional, then

   $$\dim \ker(f) + \dim \operatorname{im}(f) = \dim V$$

   Solution. True, this is the dimension theorem.

   (i) If $f : V \to W$ is a linear transformation and $\dim W < \dim V < \infty$, then f cannot be 1-1.

   Solution. True. By the dimension theorem, $\dim \ker(f) + \dim im(f) = \dim V$. Since $\operatorname{im}(f) \subset W$, $\dim \operatorname{im}(f) \leq \dim W < V$. It follows that $\dim \ker(f) > 0$, f has a nonzero kernel, so f is not 1-1 (see, for example, Theorem 2.4 in the book).

   (j) If W_1, W_2 are subspaces of a vector space V, then $W_1 \subset W_2$ if and only if $\dim W_1 \leq \dim W_2$.

   Solution. False. Take $V = \mathbb{R}^2$, and take W_1, W_2 to be the two axes respectively. Then their dimensions are equal, but neither contains the other.

2. (15 points)

(a) (4p) Let V be a vector space, and let $v \in V$ be a nonzero vector. Prove that for any $a \in F$, $av = 0 \iff a = 0$

Proof. The direction $\Leftarrow$ is clear, since $0v = 0$. Now suppose $av = 0$. Then if $a \neq 0$, there exists $a^{-1} \in F$ such that $a^{-1}a = 1$, in which case we would get $v = 1v = a^{-1}av = a^{-1}0 = 0$. Since v was assumed nonzero this means $a = 0$. □

(b) (4p) Let V be a vector space, and let $v \in V$ be a nonzero vector. Prove that for any $a, b \in F$, $av = bv \iff a = b$

Proof. If $av = bv$, then $av - bv = (a - b)v = 0$, but by (a) this implies that $a - b = 0$, so $a = b$. □

(c) (7p) Let $\mathbb{F}_3$ be a field with 3 elements. How many 1-dimensional subspaces are there in $\mathbb{F}_3 \times \mathbb{F}_3$? Justify your answer.

Solution. First, there are $3^2 = 9$ elements in $\mathbb{F}_3 \times \mathbb{F}_3$. Any 1-dimensional subspace is generated by any nonzero vector in it. A 1-dimensional subspace over $\mathbb{F}_3$ has 3 elements, one of which is the zero vector, so every 1-dimensional subspace over $\mathbb{F}_3$ has two possible bases. There are $9 - 1 = 8$ nonzero vectors in $\mathbb{F}_3 \times \mathbb{F}_3$, these elements can be labelled by the 1-dimensional subspaces that they generate. Since every nonzero vector generates a unique 1-dimensional subspace, and every 1-dimensional subspace has exactly two generators, there must be 4 such subspaces.

3. (20 points) Let $f : V \to W$ be a linear transformation. In the following two proofs, do not invoke any theorems about linear transformations. You may of course use the definition of a linear transformation, and the extension that $f(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i f(x_i)$ for any $a_1, \dots, a_n \in F$ and $x_1, \dots, x_n \in V$.

(a) (10p) Suppose f is one-to-one, and $S \subset V$ is a subset. Prove that S is linearly independent if and only if $f(S)$ is linearly independent. (Recall: $f(S) := \{f(s) \mid s \in S\}$)

Proof. We first prove $\Rightarrow$. Suppose S is LI, and suppose $a_1, \dots, a_n \in F, s_1, \dots, s_n \in S$ satisfy $a_1 f(s_1) + \dots + a_n f(s_n) = 0$. By linearlity, this means $f(a_1 s_1 + \dots + a_n s_n) = 0$. Since $f(0) = 0$ and f is 1-1, anything that gets mapped to 0 must be zero, so $a_1 s_1 + \dots + a_n s_n = 0$. Since S is LI, the a_i's are all 0. Since the s_i's were arbitrary, this shows that $f(S)$ is LI.

Next we prove $\Leftarrow$. Suppose $f(S)$ is LI, and let $a_1, \dots, a_n \in F, s_1, \dots, s_n \in S$ satisfy $a_1 s_1 + \dots + a_n s_n = 0$. Then $f(a_1 s_1 + \dots + a_n s_n) = a_1 f(s_1) + \dots + a_n f(s_n) = 0$. Since $f(S)$ is LI, this implies the a_i's are all 0. Since the s_i's were arbitrary, this shows that S is LI.

□

(b) (10p) Suppose $\beta = \{v_1, \dots, v_n\}$ is a basis for V and that f is both one-to-one and onto. Prove that $f(\beta)$ is a basis for W.

Proof. Since β is a basis, it is LI, so by (a), $f(\beta)$ is LI. It remains to show that $f(\beta)$ spans W. Let $w \in W$. We must show that w is in the span of $f(\beta)$. Since f is onto, $w = f(v)$ for some $v \in V$. Since β is a basis, we can write $v = a_1 v_1 + \cdots + a_n v_n$, but this shows that

$$w = f(v) = f(a_1 v_1 + \cdots + a_n v_n) = \sum_i a_i f(v_i)$$

This shows that $w \in \operatorname{Span} f(\beta)$. □

4. (25 points) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$f(x, y, z) = (x + 2y + 3z, 5x + 5z, -x + y)$$

Let $\beta = ((1,0,0),(0,1,0),(0,0,1))$ be the standard ordered basis of $\mathbb{R}^3$. Let γ be the ordered basis $((1,0,0),(0,1,2),(1,1,1))$.

(a) (5p) Find the matrix of f relative to the standard ordered basis β. This matrix is denoted $[f]_\beta^\beta$ (or sometimes just $[f]_\beta$ for short).

Solution. This just amounts to unraveling the definition of $[f]_\beta^\beta$. In this case, since we're working relative to the standard basis, this is just the matrix whose jth column is the column vector $f(e_j)$. Thus the matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) (2p) Compute the matrix product $[f]_\beta^\beta \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $x, y, z \in \mathbb{R}$. Your answer should be a matrix whose entries are in terms of x, y, z.

Solution. The product is just

$$\begin{bmatrix} x + 2y + 3z \\ 5x + 5z \\ -x + y \end{bmatrix}$$

(c) (5p) Find the second column of the matrix $[f]_\beta^\gamma$.

Solution. The second column is the coordinate vector of $f(e_2)$ relative to the basis γ. From the definition of f, we have $f(e_2) = f(0,1,0) = (2,0,1)$. We wish to express this as a linear combination of the vectors in γ. In other words, we want to find coefficients a_1, a_2, a_3 such that

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

This corresponds to the matrix equation:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

or equivalently, the system of linear equations

$$\begin{aligned} a_1 + a_3 &= 2 \\ a_2 + a_3 &= 0 \\ 2a_2 + a_3 &= 1 \end{aligned}$$

Using techniques from Math 250 (i.e., solving the linear system) we find that the second column is $(a_1, a_2, a_3) = (3, 1, -1)$.

(d) (5p) Find a basis for $\operatorname{im}(f)$. Explain why it is a basis.

Solution. The image of f is the span of $f(\beta)$, which is $v_1 := (1,5,-1), v_2 := (2,0,1), v_3 := (3,5,0)$. Note that these are just the columns of the matrix $[f]_\beta$. Since a basis is just a minimal spanning set, there is a subset of $\{v_1, v_2, v_3\}$ which is a basis. We just have to find this subset. Since $\operatorname{im}(f) \subset \mathbb{R}^3$, it is at most 3 dimensional. It's obvious that v_1, v_2, v_3 are not all scalar multiples of each other, so they span something which is at least 2-dimensional. In other words, $\operatorname{im}(f) = 2$ or 3, and $\operatorname{im}(f) = 3$ if and only if $\{v_1, v_2, v_3\}$ are LI. To check this, consider the equation

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

Writing it out as in part (c), we find that this is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently the linear system

$$\begin{aligned} a_1 + 2a_2 + 3a_3 &= 0 \\ 5a_1 + 5a_3 &= 0 \\ -a_1 + a_2 &= 0 \end{aligned}$$

Solving this, we find that the general solution is $a_1 = a_2 = -a_3$. Thus, there are nontrivial solutions (e.g., $(1,1,-1)$), so $\{v_1, v_2, v_3\}$ are not linearly independent. Thus, $\operatorname{rank}(f) = 2$, so a basis of $\operatorname{im}(f)$ is given by a size 2 subset of $\{v_1, v_2, v_3\}$. For example, $\{v_1, v_2\}$ give a basis, since it is a linearly independent set of size 2 inside the 2-dimensional space $\operatorname{im}(f)$.

(e) (5p) Find a basis for $\ker(f)$. Explain why it is a basis.

Solution. From part (d), $\ker(f)$ is the set $\{(a, a, -a) \mid a \in \mathbb{R}\} \subset \mathbb{R}^3$. This is a 1-dimensional space, so any nonzero vector gives a basis. For example, you can take $(1, 1, -1)$.

(f) (3p) Is f one-to-one? Is f onto? Why?

Solution. f has a nonzero kernel, so it is not 1-1. Since $\dim \operatorname{im}(f) = 2$, $\operatorname{im}(f)$ cannot be equal to $\mathbb{R}^3$, so f cannot be onto.

5. (10 points) Let V, W be nonzero finite dimensional vector spaces. Recall that $\mathcal{L}(V, W)$ denotes the space of linear transformations $V \to W$. Let $v \in V$ be a nonzero vector. Let $\mathcal{L}_v \subset \mathcal{L}(V, W)$ be the subset

$$\mathcal{L}_v := \{f \in \mathcal{L}(V, W) \mid f(v) = 0\}$$

(a) (3p) Show that $\mathcal{L}_v \subset \mathcal{L}(V, W)$ is a subspace.

Proof. Clearly $\mathcal{L}_v$ contains the zero linear transformation. Now, if $f, g \in \mathcal{L}(v)$, then $(f+g)(v) = f(v) + g(v) = 0$, so $f + g \in \mathcal{L}(v)$. Similarly if $a \in F$, then $(af)(v) = af(v) = a \cdot 0 = 0$, so $af \in \mathcal{L}(v)$. This shows that $\mathcal{L}_v$ is a subspace. □

(b) (7p) Show that $\dim \mathcal{L}_v = \dim \mathcal{L}(V, W) - \dim(W)$. (Hint: Consider the map $e_v : \mathcal{L}(V, W) \to W$ sending $f \mapsto f(v)$. Show that e_v is linear. What is its image? kernel?)

Proof. Consider the map $e_v : \mathcal{L}(V, W) \to W$. This map is linear since $e_v(f + g) = (f + g)(v) = f(v) + g(v) = e_v(f) + e_v(g)$, and $e_v(af) = (af)(v) = af(v) = ae_v(f)$. Its kernel is obviously $\mathcal{L}(v)$. We claim that its image is W (i.e., e_v is surjective). Indeed, for any $w \in W$, pick a basis $v_1, \ldots, v_n$ for V. Then there is a linear transformation f that sends each $v_i \mapsto w$ (for example, Theorem 2.6 in the book guarantees this). It follows that $e_v(f) = f(v) = w$. This shows that $\operatorname{im}(e_v) = W$. By the dimension theorem, we have

$$\dim \ker(e_v) + \dim \operatorname{im}(e_v) = \dim \mathcal{L}(v) + \dim W = \dim \mathcal{L}(V, W)$$

as desired. □