Two base change results for rings of invariants

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1 Introduction

Many results in invariant theory are described over fields, or sometimes over $\mathbb{Z}$, even though in fact they hold over all rings. The latter is well known, though often not written down. In this short note we carefully explain how to promote two such results, which are well known (and explained) over fields, to arbitrary rings R. The key result is the universal coefficient theorem, which relates base change properties to the vanishing of a certain Tor group, and the notion of a good filtration, which will imply the vanishing of desired Tor group. We thank Wilberd van der Kallen for his patience in explaining some of these ideas to us in a mathoverflow post.

Please contact me at oxeimon[at]gmail[dot]com with any questions, comments, or mistakes.

1.1 The problem

Let k be a ring (commutative with 1). For integers $d, n \geq 1$, let $k[n]$ denote the polynomial ring on nd^2 variables corresponding to the coordinates of n-many $d \times d$ matrices. Thus $k[n]$ represents the functor

$$\begin{array}{rcl} M_{n,d} : \underline{\mathbf{Alg}}_k & \longrightarrow & \underline{\mathbf{Sets}} \\ R & \mapsto & M_d(R) \end{array}$$

where $M_d(R)$ denotes the set of $d \times d$ matrices with coefficients in the k-algebra R. For $A \in M_d(A)$, let $c_k(A)$ be the coefficient of T^k in the characteristic polynomial $\det(A - TI)$. The group scheme $\mathrm{GL}_d = \mathrm{GL}_{d,k}$ acts on the functor $M_{n,d}$, and hence on $k[n]$ by simultaneous conjugation. Clearly for any product $X_{i_1} X_{i_2} \cdots X_{i_r}$ ($i_j \in \{1, \ldots, n\}$ and $r \geq 1$), the function

$$c_k(X_{i_1} X_{i_2} \cdots X_{i_r}) \in k[n]$$

is GL_d-invariant. We have the following classical theorem of invariant theory

Theorem 1.1.1 (First fundamental theorem for the invariant theory of matrices)**.** *For any ring k, $k[n]^{\mathrm{GL}_d}$ is generated as a k-algebra by the functions $c_k(X_{i_1} \cdots X_{i_r})$.*

The statement over $\mathbb{C}$ is a classical result independently obtained by Sibirski [Sib67] and Procesi [Pro76]. It was then extended to the cases $k = \mathbb{Z}$ and k any algebraically closed field by Donkin [Don92] (also see [DCP17, §15.2]). Here we describe how to deduce Theorem 1.1.1 from Donkin's results. By the universal coefficient theorem [Jan03, I, Proposition 4.18], for any flat algebraic group G over $\mathbb{Z}$, a flat G-module M and any ring k, we have an exact sequence

$$0 \longrightarrow M^G \otimes k \longrightarrow (M \otimes k)^{G_k} \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H^1(G, M), k) \longrightarrow 0 \tag{1}$$

Thus, the problem of base change is reduced to showing that $H^1(G, M)$ is $\mathbb{Z}$-flat. In fact it is possible to show that this cohomology group vanishes. We will also show a related result:

Theorem 1.1.2 (Base change for strict character varieties)**.** *Let G be a split reductive algebraic group over $\mathbb{Z}$. For an integer $n \geq 1$, let $\mathbb{Z}[G^n]$ be the coordinate ring of G^n, and let G act on $\mathbb{Z}[G^n]$ by simultaneous conjugation. Then we have*

$$H^i(G, \mathbb{Z}[G^n]) = 0 \qquad \textit{for all } i > 0$$

In particular, for any ring k, $\mathbb{Z}[G^n]^G \otimes k = k[G^n]^{G_k}$.

This result is particularly important in the theory of character varieties for representations of free groups. The connection is this - let F_n denote a free group of rank n. Then the functor $\mathrm{Hom}(F_n, G)$ is representable by G^n, and we may consider the GIT quotient $\mathrm{Hom}(F_n, G)/\!\!/G \cong (G^n)/\!\!/G = \operatorname{Spec}\mathbb{Z}[G^n]^G$ where G acts by conjugation on the target of any representation in $\mathrm{Hom}(F_n, G)$, or equivalently by simultaneous conjugation on G^n. When $G = \mathrm{GL}_d$, this quotient is called the *character variety* for degree d representations of F_n. For general G, let us call this quotient the G-character variety for G-representations of F_n (we consider only conjugation by G, not GL_d). The general theory implies that this is a categorical quotient [Ses77, Remark 8]; Theorem 1.1.2 moreover shows that the quotient commutes with arbitrary base change in the base ring.

2 First fundamental theorem of invariant theory

Here $G = \mathrm{GL}_d$ and $M = \mathbb{Z}[n]$. We wish to show that $H^1(G, M)$ is $\mathbb{Z}$-flat, or equivalently that $\mathrm{Tor}_1^{\mathbb{Z}}(H^1(G, M), \mathbb{F}_p) = 0$ for all primes p [Har10, §1, Lemma 2.1]. Since $\overline{\mathbb{F}_p}$ is faithfully flat over $\mathbb{F}_p$, it suffices to show

$$\mathrm{Tor}_1^{\mathbb{Z}}(H^1(G, M), \overline{\mathbb{F}_p}) = 0 \qquad \text{for all primes } p$$

By universal coefficients (1), it suffices to show that $\mathbb{Z}[n]^{\mathrm{GL}_d} \otimes \overline{\mathbb{F}_p} = \overline{\mathbb{F}_p}[n]^{\mathrm{GL}_{d,\overline{\mathbb{F}_p}}}$ - equivalently, that the latter is generated as an $\overline{\mathbb{F}_p}$-algebra by the functions $c_k(X_{i_1} \cdots X_{i_r})$. However this follows from Donkin's main theorem [Don92, Theorem 1].

3 Base change for G-character varieties

Here we address Theorem 1.1.2. The key idea is that of a good filtration.

3.1 Some notation

Let G be a split reductive algebraic group over a ring k and T a maximal torus.[1] Let R be the set of roots, and $R^+ \subset R$ a positive system. Let B the negative Borel subgroup, let $X(T)$ denote the weight lattice of T, and $X^+(T) \subset X(T)$ the subset which are dominant for G. A subset $\pi \subset X^+(T)$ is saturated if whenever $\lambda \in \pi$, then for any $\mu \in X^+(T)$ with $\mu \le \lambda$, then also $\mu \in \pi$. We note that $X^+(T)$ is the union of all finite saturated subsets π.

For a weight $\lambda \in X(T)$, let k_λ denote the representation of B on which T acts via λ. For any G-module M, let $H^i(M) := R^i \operatorname{Ind}_B^G M \cong H^i(G/B, \mathcal{L}(M))$, where $\mathcal{L}$ is the quasi-coherent sheaf on G/B associated to M [Jan03, I, §5.8-5.12]. For $\lambda \in X(T)$, let $H^i(\lambda) := H^i(k_\lambda)$. Let W be the Weyl group of G, and let $w_0 \in W$ be the element of longest length.

When k is a field, there is a universal highest weight module of weight λ, denoted $V(\lambda) := H^0(-w_0\lambda)^*$ (also called the Weyl module) [Jan03, II, §2.13]. Inside $V(\lambda)_{\mathbb{Q}} := \operatorname{Ind}_{B_{\mathbb{Q}}}^{G_{\mathbb{Q}}}(-w_0\lambda)^*$, there is a $\mathbb{Z}$-lattice, denoted $V(\lambda)_{\mathbb{Z}}$, which is $G_{\mathbb{Z}}$-stable, and satisfies $V(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} K = V(\lambda)_K$ for any field K [Jan03, II, §8.3]. Accordingly for arbitrary rings A, define $V(\lambda)_A := V(\lambda) \otimes_{\mathbb{Z}} A$.

We say that a G-module M has a *good filtration* if there exists an ascending filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ with $M = \bigcup_{i \ge 0} M_i$ and each V_i/V_{i-1} is isomorphic to some $H^0(\lambda_i)$ with $\lambda_i \in X^+(T)$.

3.2 The argument

The following results are key.

Lemma 3.2.1 ([Jan03, II, Lemma B.9])**.** *Suppose k is a principal ideal domain. Let M be a G-module that is free of finite rank over k. Then the following properties are equivalent:*

[1] In fact any split reductive algebraic group is defined over $\mathbb{Z}$. See the references in [Jan03, II, Introduction to §1].

(i) M *has a good filtration.*

(ii) $\mathrm{Ext}^i_G(V(\lambda), M) = 0$ *for all* $\lambda \in X^+(T)$ *and all* $i > 0$.

(iii) $\mathrm{Ext}^1_G(V(\lambda), M) = 0$ *for all* $\lambda \in X^+(T)$.

(iv) For each maximal ideal $\mathfrak{m}$ *in* k*, the* $G_{k/\mathfrak{m}}$*-module* $M \otimes k/\mathfrak{m}$ *has a good filtration.*

Here is the analog of Donkin's theorem [Don92, Theorem 1]:

Proposition 3.2.2. *Let* k *be a field. For an integer* $n \geq 1$*, let* G *act on* $k[G^n]$ *by simultaneous conjugation. Then* $k[G^n]$ *has a good filtration.*

Proof. First, let $k[G]_{l,r}$ be $k[G]$ with the $G \times G$-module structure given by the left and right regular representations. Then by [Jan03, II, Proposition 4.20], $k[G]_{l,r}$ has a good filtration with factors $H^0(\lambda) \otimes H^0(-w_0\lambda)$ where each dominant weight λ appearing once. Then $k[G]$ with the conjugation action is obtained by restricting the $G \times G$ action on $k[G]_{l,r}$ via the diagonal embedding $\Delta : G \hookrightarrow G \times G$ (remember an inverse is required in either the left or right regular representations). On the other hand tensor products of G-modules with good filtrations also admit good filtrations [Jan03, II, Proposition 4.21], so each $H^0(\lambda) \otimes H^0(-w_0\lambda)$ has a good filtration. Thus we conclude that $k[G]$ has a good filtration as a G-module acting by conjugation (also see [Jan03, II, Remark 4.21]. Since $k[G^n] = k[G]^{\otimes n}$, we find that $k[G^n]$ has a good filtration as desired. ☐

Working over $k = \mathbb{Z}$, since G/B is a smooth proper $\mathbb{Z}$-scheme[2], when $\lambda = 0$, $V(0) = H^0(\mathbb{Z})^* = H^0(G/B, \mathcal{O}_{G/B})^* = \mathbb{Z}$. Thus, if $\mathbb{Z}[G^n]$ has a filtration with each filtered piece finite free and having a good filtration, then by applying Lemma 3.2.1 to the filtered pieces, we would find that $H^i(G, \mathbb{Z}[G^n]) = 0$ for all $i > 0$, as desired.

To describe these pieces, we will need the truncated submodules $O_\pi(\mathbb{Z}[G]) := \mathbb{Z}[G] \cap O_\pi(\mathbb{Q}[G])$, where $\pi \subset X^+(T)$ is a subset [Jan03, II, §A.1, A.24, B.7]. These submodules are free $\mathbb{Z}$-modules, stable under G (acting by conjugation), and are finite rank if π is finite [Jan03, II, §A.15-16]. In particular, for any list of finite saturated subsets $\pi_1, \pi_2, \ldots, \pi_n \subset X^+(T)$,

$$O(\pi_1, \ldots, \pi_n) := \bigotimes_{i=1}^{n} O_{\pi_i}(\mathbb{Z}[G]) \quad \subset \quad \mathbb{Z}[G]^{\otimes n} \; = \; \mathbb{Z}[G^n]$$

is a finite free G-submodule of $\mathbb{Z}[G]$. We have $O_{X^+(T)}(\mathbb{Q}[G]) = \mathbb{Q}[G]$ [Jan03, II, §A.1], so setting we find that $\mathbb{Z}[G] = \bigcup_\pi O_\pi(\mathbb{Z}[G])$ as π ranges over finite saturated subsets of $X^+(T)$, and hence $\mathbb{Z}[G^n] = \mathbb{Z}[G]^{\otimes n} = \bigcup_{(\pi_1,\ldots\pi_n)} O(\pi_1, \ldots, \pi_n)$. It remains to show that each $O(\pi_1, \ldots, \pi_n)$ admits a good filtration. By Lemma 3.2.1, it suffices to check this over fields k. Indeed, for a field k, since $k[G]$ has a good filtration by Proposition 3.2.2, the same is true of $O_\pi(k[G])$ (use [Jan03, II, Lemma A.15] and [Jan03, II, Proposition 4.21]), and hence the same is true of the tensor product $O(\pi_1, \ldots, \pi_n)_k$ for any sequence $\pi_1, \ldots, \pi_n$ of finite saturated subsets. This completes the proof.

4 References

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[2] That G/B is fppf over $\mathbb{Z}$, see [Jan03, I, §5.6-5-7]. Using this one can check smoothness and properness on geometric fibers, which follow from the classical theory