$T: V \to V$, $\dim V < \infty$, λ eigenvalue

$K_\lambda \supseteq E_\lambda$ is T-invariant $\Rightarrow$ it is also $f(T)$-invariant for any poly $f(t)$

$K_\lambda = \bigcup_{p \geq 1} N((T-\lambda I)^p)$

- $(T-\mu I)_{K_\lambda}$ is an isom $K_\lambda \xrightarrow{\sim} K_\lambda$ if $\mu \neq \lambda$
- $K_\lambda \cap K_\mu = 0$ if $\mu \neq \lambda$
- If χ_T is split w/ roots $\lambda_1, \dots, \lambda_k$, then if β_{λ_i} is any basis for K_{λ_i}, then $\beta := \beta_{\lambda_1} \cup \beta_{\lambda_2} \cup \dots \cup \beta_{\lambda_k}$ is a basis for V
- $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_k} \Rightarrow K_{\lambda_i} \cap \left(\sum_{j \neq i} K_{\lambda_j}\right) = 0 \quad \forall i$

$$\Rightarrow [T]_\beta = \begin{bmatrix} [T_{K_{\lambda_1}}]_{\beta_{\lambda_1}} & & O \\ & \ddots & \\ O & & [T_{K_{\lambda_k}}]_{\beta_{\lambda_k}} \end{bmatrix}$$

Def If x is a generalized λ-evector, s.t. p is the min. pos. int w/ $U^p x = 0$ $(U = T - \lambda I)$, then

$$C_x = \{\underbrace{U^{p-1}x}_{\text{initial vector}}, U^{p-2}x, \dots, Ux, \overset{\text{end vector}}{x}\}$$

$|C_x| = p =$ "length of the cycle C_x"

Thm If $C_1, \dots, C_r$ are cycles of gen. λ-evectors s.t. the initial vectors are distinct and L.I., then $\bigcup_{i=1}^r C_i$ is also L.I. (and $C_i \cap C_j = \phi$ for $i \neq j$)

Q Could $\exists v \in C_x$ which is a μ-eigenvector? (for some $\mu \neq \lambda$)

A No! (since $K_\lambda \cap K_\mu = 0$)

Thm K_λ admits a basis consisting of a union of cycles.

$$[T_{\langle C_x \rangle}]_{C_x} = \underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & O \\ & & \lambda & \ddots \\ O & & & 1 \\ & & & \lambda \end{bmatrix}}_{\text{Jordan block}} \qquad [T_{K_\lambda}]_{\beta_\lambda} = \begin{bmatrix} [T_{\langle C_1 \rangle}]_{C_1} & & O \\ & \ddots & \\ O & & [T_{\langle C_r \rangle}]_{C_r} \end{bmatrix}$$

$\left(\beta_\lambda = \bigcup_{i=1}^r C_i\right)$

$\chi\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} = (-t)^3 = -t^3$

Def The minimal polynomial of T is a $\overset{\text{leading coeff. is } = 1}{\text{monic}}$ polynomial $p(t)$ of lowest positive degree s.t. $p(T) = 0$

Note If $p(t)$ divides $g(t)$, then $g(t) = q(t)p(t)$ for some poly. $q(t)$

$\Rightarrow g(T) = q(T)\underbrace{p(T)}_{0} = 0$

$$\left(\begin{array}{l} q(T)p(T) := q(T) \circ p(T) \\ V \xrightarrow{p(T) = 0} V \xrightarrow{q(T)} V, \quad q(T)p(T) = p(T)q(T) \\ v \longmapsto 0 \longmapsto 0 \end{array} \right)$$

Thm (7.12 in the book) (§7.3)

ⓐ If $g(t)$ is a poly s.t. $g(T) = 0$, then $p(t)$ divides $g(t)$ (for any minimal poly. $p(t)$)

ⓑ The min. poly is unique.

(ⓐ $\Rightarrow p(t)$ divides $\chi_T(t)$)

Pf ⓐ $g(T) = 0$, then by the division algorithm, $\exists q(t), r(t)$ s.t.

$$g(t) = q(t)p(t) + r(t)$$

and $\deg r(t) < \deg p(t)$.

$$\begin{array}{c} 13 = 4 \cdot 3 + 1 \\ q \quad p \quad r \\ r < p = 3 \end{array}$$

$(t-1)(t-2)^2$

$t-1$ or $t-2$ or $(t-1)(t-2)$ or $(t-2)^2$ or $(t-1)(t-2)^2$

But
$$0 = g(T) = q(T)\underbrace{p(T)}_{0} + r(T) = r(T)$$

$\Rightarrow r(T) = 0$ (but $\deg r(t) < \deg p(t)$)

by minimality of $\deg p(t)$, this $\Rightarrow r(t) = 0$, ie $p(t)$ divides $g(t)$

(b) If $p_1(t)$, $p_2(t)$ are min. poly's, then by (a), $p_1(t)$ divides $p_2(t)$, so $p_2(t) = g(t)p_1(t)$, but $\deg p_1(t) = \deg p_2(t) \Rightarrow \deg g(t) = 0$

ie $\underset{\text{leading coeff}=1}{p_1(t)} = \underbrace{c\,p_2(t)}_{\text{leading coeff}=c}$ for some $c \in F$

But $p_1(t)$, $p_2(t)$ are monic $\Rightarrow c = 1 \Rightarrow p_1(t) = p_2(t)$

Ex $J = \left(\begin{array}{ccc|cc} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & O \\ 0 & 0 & 2 & & \\ \hline & O & & 2 & 1 \\ & & & 0 & 2 \end{array}\right) \rightsquigarrow \chi_J = (2-t)^5$ eigenvalues: 2 (only eigenvalue)

$K_2 = \mathbb{R}^5$ dot diagram is

$Q = \begin{pmatrix} 0 & 1 & -18 & 0 & 0 \\ -1 & 0 & 4 & 2 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & 0 & 1 \\ 0 & -1 & 19 & 0 & 0 \end{pmatrix}$

What is a $\underbrace{\text{Jordan canonical basis}}_{\text{JCB}}$ for J? Ans: standard basis! (w/ cycles C_{e_3}, C_{e_5}
$\quad \{e_1, e_2, e_3\}$ $\{e_4, e_5\}$

$Je_1 = 2e_1$ $\qquad (J-2I)e_1 = 0$

$Je_2 = 2e_2 + e_1$ $\qquad (J-2I)e_2 = 2e_2 + e_1 - 2e_2 = e_1$

$Je_3 = 2e_3 + e_2$ $\qquad (J-2I)e_3 = 2e_3 + e_2 - 2e_3 = e_2$

$\{\underbrace{(J-2I)^2e_3}_{e_1}, \underbrace{(J-2I)e_3}_{e_2}, e_3\}$

What is a JCB for QJQ^{-1}? Ans: the columns of Q
Exercise: check this!

Ex $B = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 \\ -7 & 2 & 0 & 2 & -6 \\ 6 & 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$ (in fact, $B = QJQ^{-1}$, but we won't use this)

$\chi_B = (2-t)^5$ only eigenvalue is 2

Compute JCF, JCB
also find min. poly.

Recall First r rows of dot diagram is a basis for $N(U^r)$

(1) Find dot diagram $\quad U = B - \underset{\text{evalue}}{2}I$

Compute $\dim N(U) = 2 \rightarrow$ 1st row is • • (so DD is either : : or ⋮ ·)

Compute $\dim N(U^2) = 4 \Rightarrow$ 1st 2 rows has 4 dots

$\Rightarrow$ DD is ⋮ :

$\Rightarrow \text{JCF} = \left(\begin{array}{ccc|cc} 2 & 1 & 0 & & \\ & 2 & 1 & & \\ & & 2 & & \\ \hline & & & 2 & 1 \\ & & & & 2 \end{array}\right)$

$U\uparrow$ $U\uparrow$ $U\uparrow$ $\quad \uparrow U$ $\uparrow U$

(2) Find cycle of length 3

$\Rightarrow$ minpoly divides χ_B

$\Rightarrow$ minpoly $= \pm(2-t)^r$ for some r

Want: maximum r s.t. $(2I-B)^r = 0$

equiv $(B-2I)^r = 0$

equiv $(B-2I)^r$ sends a basis of K_2 to 0

ie U^r sends a basis of K_2 to 0

$\Rightarrow r = 3$ (look at dot diagram for K_2)

$\Rightarrow$ minpoly $-(2-t)^3 = (t-2)^3$