$T: V \to V$, $\dim V < \infty$, λ eigenvalue. We know

① K_λ is T-invariant, and hence $f(T)$-inv. for any poly. $f(t)$ (e.g., K_λ is $(T-aI)$-invariant)

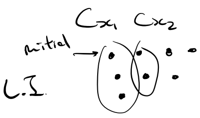
② $E_\lambda \subseteq K_\lambda \cong F^{m_\lambda}$ In particular $1 \le \dim E_\lambda \le \dim K_\lambda = m_\lambda$

③ If χ_T splits w/ roots $\lambda_1, \dots, \lambda_k$, then $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

<u>Def</u> The cycle of gen. λ-evectors generated by a gen. evector x is

$$C_x = \{\underbrace{(T-\lambda I)^{p-1}x}_{\text{initial}}, \dots, (T-\lambda I)x, \underbrace{x}_{\text{end}}\}$$

where p is the least pos. integer w/ $(T-\lambda I)^p x = 0$



④ If $C_{x_1}, \dots, C_{x_r}$ are cycles w/ LI initial vectors, then $C_{x_1} \cup \dots \cup C_{x_r}$ is L.I.

⑤ K_λ has a basis consisting of a union of cycles.

⑥ If χ_T split w/ roots $\lambda_1, \dots, \lambda_k$, and β_{λ_i} is a basis for K_{λ_i}, $\beta := \bigcup_i \beta_{\lambda_i}$, then

$$[T]_\beta = \begin{bmatrix} [T_{K_{\lambda_1}}]_{\beta_{\lambda_1}} & & O \\ & \ddots & \\ O & & [T_{K_{\lambda_k}}]_{\beta_{\lambda_k}} \end{bmatrix}$$

If χ_T split, then Jordan canonical form of T exists and is unique up to the order of the eigenvalues.

⑦ If β_λ is a union of cycles $C_{x_1}, \dots, C_{x_r}$, then

$$[T_{K_\lambda}]_{\beta_\lambda} = \begin{bmatrix} [T_{\langle C_{x_1}\rangle}]_{C_{x_1}} & & O \\ & \ddots & \\ O & & [T_{\langle C_{x_r}\rangle}]_{C_{x_r}} \end{bmatrix} \quad \text{where} \quad [T_{\langle C_{x_i}\rangle}]_{C_{x_i}} = \overbrace{\begin{bmatrix} \lambda & 1 & & O \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ & & & \ddots & 1 \\ O & & & & \lambda \end{bmatrix}}^{\text{Jordan block}}$$

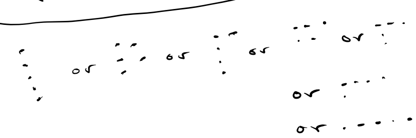
⑧ minpoly_T is the monic poly of minimum positive degree satisfying $\text{minpoly}_T(T) = 0$ (note: $\chi_T(T) = 0$)

This is unique, and if $g(t)$ is any poly satisfying $g(T) = 0$, then $\text{minpoly}_T(t)$ divides $g(t)$

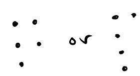
$\Rightarrow \text{minpoly}_T$ is the smallest monic divisor of χ_T satisfying $\text{minpoly}_T(T) = 0$

<u>Thm</u> The 1st r rows of the dot diagram is a basis for $N((T-\lambda I)^r)$

<u>Ex</u> $B = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 \\ \vdots & & & & \end{pmatrix} \in M_5(F)$ $\quad \dim V = 5 \Rightarrow K_2 = V$ is 5-dim. $\quad \chi_B = (2-t)^5 = -(t-2)^5$ $\quad U = B - 2I$



Suppose $\dim N(U) = 2$, then dot diagram is



" $\dim N(U^2) = 4$, then " " "

$\Rightarrow \dim N(U^3) = 5$

$\Rightarrow \text{minpoly}_B = (t-2)^3$

$$\Rightarrow \text{JCF is} \begin{bmatrix} 2 & 1 & 0 & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 2 & 1 \\ & & & & 2 \end{bmatrix}$$

Q) Find a JCB for B. (Remember $N(U^{i+1}) \supseteq N(U^i)$)

① "Find 1st column"

1a - Find a $v_1 \in \overset{\dim 5}{N(U^3)} - \overset{\dim 4}{N(U^2)}$ (Guess and check)

1b - Can choose 1st column to be represented by

$$C_{v_1} = \{U^2 v_1, Uv_1, v_1\}$$

② "Find 2nd column"

2a - Find a $v_2 \in \overset{\dim 4}{N(U^2)} - \overbrace{(N(U) + \mathrm{Span}\{Uv_1\})}^{\dim 3}$ (Guess and check)

2b - Can choose 2nd col. to be rep. by

$$C_{v_2} = \{Uv_2, v_2\}$$

Claim: $C_{v_1} \cup C_{v_2}$ is a JCB

$$= \left\{\begin{matrix} U^2v_1 & Uv_2 \\ Uv_1 & v_2 \\ v_1 & \end{matrix}\right\}$$

Q) Why must $C_{v_1} \cup C_{v_2}$ be LI?

By ④, it suffices to show that $\{U^2v_1, Uv_2\}$ is LI

Know: $v_2 \notin \mathrm{Span}\{Uv_1\} + N(U)$ ⊛

Suppose $aU^2v_1 + bUv_2$ for some $a, b \in \mathbb{R}$ (WTS $a=b=0$)

$$U(aUv_1 + bv_2) = 0 \Rightarrow aUv_1 + bv_2 \in N(U)$$

$$\Rightarrow bv_2 = \underbrace{-aUv_1}_{\in \mathrm{Span}\{Uv_1\}} + \underbrace{(aUv_1 + bv_2)}_{\in N(U)}$$

$$\overset{⊛}{\Rightarrow} b = 0 \Rightarrow a = 0$$

This shows that U^2v_1, Uv_2 are LI, as desired.

Ex) $V = \{\text{polynomials in } \overbrace{2 \text{ variables}}^{x,y} \text{ of degree} \leq 2\}$

$= \mathrm{Span}\underbrace{\{1, x, y, x^2, y^2, xy\}}_{\alpha \text{ (basis of } V)}$

Let $T = \frac{\partial}{\partial x}$, so $T(a + bx + cy + dx^2 + ey^2 + fxy) = b + 2dx + fy$

T^2: $\cancel{1} \mapsto 0$, $\cancel{x} \mapsto 0$, $\cancel{y} \mapsto 0$, $x^2 \mapsto 2$, $\cancel{y^2} \mapsto 0$, $xy \mapsto 0$

$$[U^2]_\alpha = [T^2]_\alpha = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ & & & 0 & & \\ & & & 0 & & \\ & & & 0 & & \\ & & & 0 & & \\ & & & 0 & & \end{bmatrix}$$

$$[T]_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & & & 1 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \end{bmatrix}$$

$\chi_T(t) = (-t)^6 = t^6 \Rightarrow$ only eigenvalue is 0, so $U = T - 0I = T$

① $\dim N(U) = 3$

② $\dim N(U^2) = 5$ $\Rightarrow$ dot diagram is

$$\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{matrix}$$

$\Rightarrow$ JCF is

$$\begin{pmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Next, find JCB

① Find 1st column

Find $f_1 \in N(U^3) - N(U^2)$, can take $f_1 = x^2 \Rightarrow C_{f_1} = \{2, 2x, x^2\}$

② Find 2nd col.

Find $f_2 \in N(U^2) - \underbrace{(\underbrace{N(U)}_{\text{Span}\{1, y, y^2\}} + \text{Span}\{2x\})}_{\text{Span}\{1, y, y^2, x\}}$ Can take $f_2 = xy \Rightarrow C_{f_2} = \{y, xy\}$

③ Find 3rd col. Can take $f_3 = y^2$ $\Rightarrow \{\underbrace{2, 2x, x^2}_{C_{f_1}}, \underbrace{y, xy}_{C_{f_2}}, \underbrace{y^2}_{C_{f_3}}\}$ is a JCB

Q| What is minpoly$_T$? minpoly$_T(t) = t^3$!

Note: minpoly$_T$ divides $\chi_T = t^6$

$T - 0I = T$ only kills 1st row of dot diagram

$(T-0I)^2 = T^2$ " " 2nd " " "

$(T-0I)^3 = T^3$ kills everything!