Katz Modular Forms

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August 25, 2020

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1 Introduction

The goal of these notes is to provide an account of the equivalence between Katz modular forms over $\mathbb{C}$ (e.g., sections of the Hodge bundle over certain moduli stacks of elliptic curves with level structures) and classical modular forms defined as holomorphic functions on the upper half plane $\mathcal{H}$. This result, while essentially "standard", does not (to my knowledge) seem to have a self-contained reference in the literature. These notes are my attempt at providing such a reference, in a way which works for any finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (not necessarily congruence!). These notes were written mostly for my own benefit. There may be mistakes.

The main result amounts to a GAGA argument, which leads us to work with proper schemes/stacks. In §1-6, I try to provide a well-referenced account of the equivalence between Katz and classical modular forms without shying away from stacks. In §7, I prove a version of the q-expansion principle suitable for the noncongruence setting, and deduce some arithmetic consequences.

A main reference used in these notes is the *stacks project* [Sta16], which being a work in progress, is best referenced through the use of "tags". These are sequences of 4 alphanumerics, which can be looked up here:

`https://stacks.math.columbia.edu/tag`

2 Generalities on Sheaves on Stacks

Let $\mathcal{C}$ be a site, and $p : \mathcal{X} \to \mathcal{C}$ a fibered category. We give $\mathcal{X}$ the topology inherited from $\mathcal{C}$. That is, a family $\{x_i \to x\}_{i\in I}$ is a covering in $\mathcal{X}$ iff its image in $\mathcal{C}$ is a covering family. Thus we may speak of sheaves on $\mathcal{X}$ (of sets, groups, rings, or any "algebraic structure" as described in [Sta16] 00YR).

2.1 Morphisms of stacks defining morphisms of topoi

Now let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of fibered categories over $\mathcal{C}$, each given the topologies inherited from $\mathcal{C}$. Then, f is a continuous and cocontinuous functor ([Sta16] 06NW), and we have induced maps of topoi:

$$(f_*, f^{-1}) : \underline{\mathbf{Sh}}(\mathcal{X}) \to \underline{\mathbf{Sh}}(\mathcal{Y})$$

given as follows. If $\mathcal{G} \in \underline{\mathbf{Sh}}(\mathcal{Y})$, then the inverse image $f^{-1}(\mathcal{G})$ is a sheaf on $\mathcal{X}$ given by

$$(f^{-1}(\mathcal{G}))(x) := \mathcal{G}(f(x)) \qquad x \in \mathcal{X}$$

This formula defines a sheaf because f is continuous[1]. For a sheaf $\mathcal{F} \in \underline{\mathbf{Sh}}(\mathcal{X})$, the direct image $f_*\mathcal{F}$ is given by

$$f_*\mathcal{F}(y) := \varprojlim_{(x,\psi)\in {}_V I^{\mathrm{opp}}} \mathcal{F}(x) \qquad \text{for any } y \in \mathcal{Y}$$

where ${}_V I$ is the category whose objects are (x, ψ) with $x \in \mathcal{X}$ and $\psi : f(x) \to y$ a morphism in $\mathcal{Y}$, and a morphism $(x, \psi) \to (x', \psi')$ is a morphism $\alpha : x \to x'$ such that $\psi' \circ f(\alpha) = \psi$. This formula defines a sheaf because f is cocontinuous[2].

Remark 2.1.1. We note that the definitions above seem opposite to what one might expect. For example, if $\mathcal{C} = \underline{\mathbf{Sch}}$, and $\mathcal{X}, \mathcal{Y}$ are representable by schemes X, Y, given a sheaf $\mathcal{G}$ on Y, one expects that f_* should be simple to define (no limits involved), and f^{-1} should be more complicated - for example, the usual formula for $f^{-1}G$ where G is a sheaf on the small Zariski site Y_{Zar} is:

$$f^{-1}G(U) = \varinjlim_{V\supset f(U)} G(V) \qquad G \in \underline{\mathbf{Sh}}(Y_{\mathrm{Zar}}) \tag{1}$$

[1]This formula is denoted f^p in [Sta16] 00WU, 06NW, and is equal to f^s because f is continuous, and hence sends sheaves to sheaves.

[2]This is also denoted ${}_p f$ in [Sta16] 06NW, and is equal to ${}_s f$ because f is cocontinuous - ie, it sends sheaves to sheaves.

as V ranges over opens. The fact that f_* is simple when $\mathcal{X}, \mathcal{Y}$ are schemes is actually recovered in the next section §2.2. The fact that f^{-1} is also simple can be seen as a peculiarity of sheaves on "big sites". For example, the reason for the limit in the formula (1) is because given an open immersion $U \to X$ (ie, an object of the site X_{Zar}), the composition $U \to X \to Y$ is typically not an open immersion (ie, is not in Y_{Zar}), and hence since G is defined only on Y_{Zar}, to define $(f^{-1}G)(U \to X)$, one must approximate $U \to X \to Y$ by objects of Y_{Zar}, hence the limit in (1). However, if G is actually a sheaf on a big site $(\underline{\mathbf{Sch}}/Y)_\tau$ (with some appropriate topology τ), then $U \to X \to Y$ is an object of $\underline{\mathbf{Sch}}/Y$, and hence no approximation is needed - one can simply define $(f^{-1}G)(U \to X) := G(U \to X \to Y)$. The fact that this gives a sheaf follows from the fact that the functor $f : (\underline{\mathbf{Sch}}/X)_\tau \to (\underline{\mathbf{Sch}}/Y)_\tau$ is *continuous*, which in turn follows tautologically from the definitions of inherited topologies and morphisms between fibered categories ([Sta16] Tag 06NW).

2.2 Computing pushforward

Now suppose that moreover $\mathcal{X}, \mathcal{Y}$ are fibered in groupoids over $\mathcal{C}$, and the morphism $f : \mathcal{X} \to \mathcal{Y}$ is representable. By definition, this means that for any $U \in \mathcal{C}$ and morphism $U \to \mathcal{Y}$, the fiber product $U \times_{\mathcal{Y}} \mathcal{X} = (\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ is a representable stack, represented by an object of $\mathcal{C}$, which we call $u(U)$, which comes with canonical maps to U and to $\mathcal{X}$. This association $(U \to \mathcal{Y}) \mapsto (u(U) \to \mathcal{X})$ can be made functorial, and as a result by the 2-Yoneda lemma we have a functor $\mathcal{X} \overset{u}{\leftarrow} \mathcal{Y}$ which by definition is right adjoint to f (see [Sta16] 06W7 and the ensuing discussion). In this case, the pushforward f_* can be more easily computed as:

$$f_*\mathcal{F}(y) = \mathcal{F}(u(y)) \qquad \text{for any } y \in \mathcal{Y} \text{ ([Sta16] 06W8, 00XW)}$$

2.3 The structure sheaf

Suppose $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$ making $(\mathcal{C}, \mathcal{O})$ into a ringed site. For example, if $\mathcal{C} = (\underline{\mathbf{Sch}}/S)_{\acute{e}t}$, then we can give it the structure of a ringed site by taking the structure sheaf $\mathcal{O} = \mathcal{O}_S$ to be given by the rule $\mathcal{O}(U \to S) := \Gamma(U, \mathcal{O}_U)$ where $\mathcal{O}_U$ is the standard structure sheaf on the scheme U, with the obvious restriction maps. If $\mathcal{C} = \underline{\mathbf{An}}_{\acute{e}t}$ (the category of complex analytic spaces equipped with the etale topology), then its structure sheaf $\mathcal{O}$ is given by the same formula $\mathcal{O}(U) := \Gamma(U, \mathcal{O}_U)$ for any $U \in \underline{\mathbf{An}}$, where here $\mathcal{O}_U$ is the sheaf of holomorphic functions on U.

The site $\mathcal{C}$ can be viewed as the final object in the category of stacks over $\mathcal{C}$, and the structure morphism $p : \mathcal{X} \to \mathcal{C}$ can be viewed as a morphism to this final object. Thus, as above we may form the pullback $p^{-1}\mathcal{O}$, and we define the structure sheaf $\mathcal{O}_{\mathcal{X}}$ of $\mathcal{X}$ to be:

$$\mathcal{O}_{\mathcal{X}} := p^{-1}\mathcal{O} \qquad \text{given by} \qquad \mathcal{O}_{\mathcal{X}}(x) := \mathcal{O}(p(x)) \qquad \text{(c.f. [Sta16] 06TU)}$$

More precisely, $\mathcal{O}_{\mathcal{X}}$ as a functor $\mathcal{X} \to \underline{\mathbf{Rings}}$ is the composition $\mathcal{X} \overset{p}{\to} \mathcal{C} \overset{\mathcal{O}}{\to} \underline{\mathbf{Rings}}$.

By our definition of inverse image sheaves, if $f : \mathcal{X} \to \mathcal{Y}$ is any morphism of fibered categories over the ringed site $\mathcal{C}$ with structure morphisms $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$, then we have:

$$p^{-1} = (q \circ f)^{-1} = f^{-1} \circ q^{-1} \quad \text{hence} \quad \mathcal{O}_{\mathcal{X}} := p^{-1}\mathcal{O} = f^{-1}q^{-1}\mathcal{O} =: f^{-1}\mathcal{O}_{\mathcal{Y}}$$

2.4 Types of modules

Let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ be a ringed site. In our case we will often want to consider a category fibered in groupoids $p : \mathcal{X} \to \mathcal{C}$ over the ringed site $(\mathcal{C}, \mathcal{O})$, which is either $\underline{\mathbf{An}}$ or $\underline{\mathbf{Sch}}/S$ with the usual structure sheaf, both equipped with the etale topology, and will set $\mathcal{S} = \mathcal{X}$ with the inherited topology, and $\mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathcal{X}} := p^{-1}\mathcal{O}$.

For a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{S}}$-modules on the site $\mathcal{S}$, and any object $s \in \mathcal{S}$, we may restrict $\mathcal{F}$ to the localized site $\mathcal{S}/s$ ([Sta16] 00XZ) by the rule

$$(\mathcal{F}|_{\mathcal{S}/s})(s' \to s) := \mathcal{F}(s')$$

If $\mathcal{S} = \mathcal{X}$, then for $x \in \mathcal{X}$, let $U := p(x)$. By the 2-Yoneda lemma ([Sta16] 004B), the object x defines a morphism $x : U \to \mathcal{X}$, and $\mathcal{O}_x := \mathcal{O}_U = x^{-1}\mathcal{O}_{\mathcal{X}} = (p \circ x)^{-1}\mathcal{O}$. Moreover, the functor p induces an equivalence of sites $\mathcal{X}/x \to \mathcal{C}/U$ ([Sta16] 0CN0).

We have the following types of modules (c.f. [Sta16] 03DE, 03DL).

- We say that $\mathcal{F}$ is *free* if $\mathcal{F}$ is isomorphic to a direct sum $\bigoplus_{i\in I}\mathcal{O}_{\mathcal{S}}$. If I is finite of cardinality r, then we say that $\mathcal{F}$ is *finite free of rank* r.
- We say that $\mathcal{F}$ is *locally free* if for every $s\in\mathcal{S}$, there is a covering $\{s_i\to s\}$ such that each restriction $\mathcal{F}|_{\mathcal{S}/s_i}$ is a free $\mathcal{O}_{s_i}$-module. If they are moreover all finite free, then we say that $\mathcal{F}$ is *finite locally free.*
- We say that $\mathcal{F}$ *has a global presentation* if there is an exact sequence

$$\bigoplus_{j\in J}\mathcal{O}\to\bigoplus_{i\in I}\mathcal{O}\to\mathcal{F}\to 0$$

  of $\mathcal{O}$-modules. If I,J are finite, then we say that $\mathcal{F}$ *has a global finite presentation.*
- We say that $\mathcal{F}$ is *quasi-coherent* if for every $s\in\mathcal{S}$, there is a covering $\{s_i\to s\}$ in $\mathcal{S}$ such that each restriction $\mathcal{F}|_{\mathcal{S}/s_i}$ is an $\mathcal{O}_{s_i}$-module which has a global presentation.
- We say that $\mathcal{F}$ is *generated by finitely many global sections* if there is an integer $r\geq 0$ and a surjection $\mathcal{O}_{\mathcal{S}}^{\oplus r}\to\mathcal{F}$.
- We say that $\mathcal{F}$ is *finite type* if for every $s\in\mathcal{S}$, there is a covering $\{s_i\to s\}$ such that each restriction $\mathcal{F}|_{\mathcal{S}/s_i}$ is an $\mathcal{O}_{s_i}$-module generated by finitely many global sections.
- We say that $\mathcal{F}$ is *coherent* if $\mathcal{F}$ is of finite type, and for every object $s\in\mathcal{S}$ and any finite set of sections $\sigma_1,\dots,\sigma_n\in\mathcal{F}(s)$, the kernel of the map $(\sigma_i):\oplus_{i=1}^n\mathcal{O}_s\to\mathcal{F}|_s$ is of finite type on the localized site $(\mathcal{S}/s,\mathcal{O}_s)$.

2.5 Functoriality for modules

Let S be a scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Then for any reasonable topology τ, the rule sending any $f:U\to S$ to $\Gamma(U,f^*\mathcal{F})$ defines a sheaf of $\mathcal{O}_S$-modules on the big site $(\underline{\mathbf{Sch}}/S)_\tau$, denoted $\mathcal{F}^a$, and similarly for the small etale or Zariski sites ([Sta16] 03DU). These sheaves $\mathcal{F}^a$ are moreover quasicoherent ([Sta16] 03DV), and the construction $\mathcal{F}\mapsto\mathcal{F}^a$ determines an *equivalence of categories* which is compatible with pullback ([Sta16] 03DX, 03LC. See 03DO for the result that pullback preserves quasicoherence):

$$\cdot^a:\underline{\mathbf{QCoh}}(S)\to\underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/S)_\tau,\mathcal{O}_S)$$

(Warning: the composition $\underline{\mathbf{QCoh}}(S)\to\underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/S)_\tau,\mathcal{O}_S)$ is *not* necessarily exact! See §2.5.1 for an example.)

Now suppose $f:\mathcal{X}\to\mathcal{Y}$ is a morphism of fibered categories over the ringed site $(\mathcal{C},\mathcal{O})$. If $\mathcal{G}$ is a sheaf of $\mathcal{O}_{\mathcal{Y}}$-modules, then for any $x\in\mathcal{X}$, $f^{-1}\mathcal{G}(x):=\mathcal{G}(f(x))$ which is a $(f^{-1}\mathcal{O}_{\mathcal{Y}})(x)=\mathcal{O}_{\mathcal{Y}}(f(x))=\mathcal{O}_{\mathcal{X}}(x)$-module, and so $f^{-1}\mathcal{G}$ is naturally an $\mathcal{O}_{\mathcal{X}}$-module. Normally, between $\underline{\mathbf{Mod}}(\mathcal{O}_{\mathcal{X}}),\underline{\mathbf{Mod}}(\mathcal{O}_{\mathcal{Y}})$, f_* is right adjoint to f^*, which is usually defined as:

$$f^*\mathcal{G}:=f^{-1}\mathcal{G}\otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}}\mathcal{O}_{\mathcal{X}}$$

but since $\mathcal{O}_{\mathcal{X}}=f^{-1}\mathcal{O}_{\mathcal{Y}}$, we find that $f^*\mathcal{G}=f^{-1}\mathcal{G}$, so $f^*=f^{-1}$ in our setting. Put another way, the fact that $f^{-1}\mathcal{O}_{\mathcal{Y}}=\mathcal{O}_{\mathcal{X}}$ implies that the pair (f_*,f^{-1}) defines a *flat* morphism of ringed topoi ([Sta16] 04JB)

$$(f_*,f^{-1}):(\underline{\mathbf{Sh}}(\mathcal{X}),\mathcal{O}_{\mathcal{X}})\to(\underline{\mathbf{Sh}}(\mathcal{Y}),\mathcal{O}_{\mathcal{Y}})$$

In particular, $f^{-1}=f^*:\underline{\mathbf{Mod}}(\mathcal{O}_{\mathcal{Y}})\to\underline{\mathbf{Mod}}(\mathcal{O}_{\mathcal{X}})$ is *exact* ([Sta16] 04JC).

2.5.1 The case of representable stacks: exactness of $f^*=f^{-1}$ on big sites

The results of 2.5 implies that if $f:X\to Y$ is *any* morphism of schemes, then $f^*=f^{-1}:\underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/Y)_\tau,\mathcal{O}_Y)\to\underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/X)_\tau,\mathcal{O}_X)$ is exact for any reasonable topology τ. Of course, the analogous statement in the case of small Zariski sites is far from true (f^* is only exact if f is flat). Let us see exactly what the difference is:

Consider the map $f:\operatorname{Spec}\mathbb{F}_2\to\operatorname{Spec}\mathbb{Z}$. Let $\mathcal{F}$ be the quasicoherent sheaf $\tilde{\mathbb{Z}}$ on $\operatorname{Spec}\mathbb{Z}$. It's clear that the map $2:\mathcal{F}\to\mathcal{F}$ is injective, but $f^*2:f^*\mathcal{F}\to f^*\mathcal{F}$ is zero. On the other hand, by the recipe given above,

$\mathcal{F}$ determines a quasicoherent sheaf $\mathcal{F}^a$ on the big étale site $(\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}$, and the map $2^a : \mathcal{F}^a \to \mathcal{F}^a$ is still an injective map of quasicoherent modules since $\mathcal{F} \mapsto \mathcal{F}^a$ is an equivalence. Applying f^*, we have a morphism $f^*(2^a) : f^*\mathcal{F}^a \to f^*\mathcal{F}^a$. Since $f^* = f^{-1}$, and taking global sections of $f^*\mathcal{F}$ over $\operatorname{Spec}\mathbb{F}_2$, we have

$$f^*\mathcal{F}^a(\operatorname{Spec}\mathbb{F}_2) = \mathcal{F}^a(\operatorname{Spec}\mathbb{F}_2 \xrightarrow{f} \operatorname{Spec}\mathbb{Z}) = \Gamma(\operatorname{Spec}\mathbb{F}_2, f^*\tilde{\mathbb{Z}}) = \mathbb{F}_2$$

and on this, $f^*(2^a)$ is again given by multiplication by 2, and hence is the zero map, so $f^*(2^a)$ is not injective! At first this seems to contradict the stated exactness of $f^* = f^{-1}$ on $\mathcal{O}$-modules and the injectivity of 2^a.

The fix is to note that the functor $\mathcal{F} \mapsto \mathcal{F}^a$ only gives an equivalence $\underline{\mathbf{QCoh}}(\operatorname{Spec}\mathbb{Z}) \to \underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O})$. However, the inclusion $\underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O}) \hookrightarrow \underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O})$ is certainly not an equivalence, and moreover it is not even left exact[3]! In particular, the map $2^a : \mathcal{F}^a \to \mathcal{F}^a$ is only injective in the category $\underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O})$ in the sense that it's quasicoherent kernel is trivial, but not injective in $\underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O})$. Indeed, $\operatorname{Spec}\mathbb{F}_2 \to \operatorname{Spec}\mathbb{Z}$ is an object of $(\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}$, and hence the same computation as above shows that $2^a : \mathcal{F}^a \to \mathcal{F}^a$ has a nontrivial $\mathcal{O}$-module kernel[4].

This example illustrates that while $f^* = f^{-1}$ is exact on big sites, producing injective maps is "more difficult". The fact that $f : X \to Y$ (and more generally $T \to X \xrightarrow{f} Y$) is an object of $(\underline{\mathbf{Sch}}/Y)_{ét}$ implies that an injective morphism of $\mathcal{O}$-modules on $(\underline{\mathbf{Sch}}/Y)_{ét}$ must by definition be injective on all classical (scheme-theoretic) pullbacks. This exactly excludes the morphisms which fail to remain injective after applying a classical pullback, which is "how" $f^* = f^{-1}$ manages to be exact on $\mathcal{O}$-modules!

Note that the equivalence $\cdot^a : \underline{\mathbf{QCoh}}(X) \to \underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/X)_{ét}, \mathcal{O}_X)$ and its compatibility with pullback implies that $f^* = f^{-1}$ is *not* generally left exact as a functor $\underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/Y)_{ét}, \mathcal{O}_Y) \to \underline{\mathbf{QCoh}}((\underline{\mathbf{Sch}}/X)_{ét}, \mathcal{O}_X)$! It is exact precisely when the usual pullback $f^* : \underline{\mathbf{QCoh}}(Y) \to \underline{\mathbf{QCoh}}(X)$ is exact, ie, when f is flat.

2.6 Analytic spaces and stacks

We follow the definitions of [Hal14]. Let $\underline{\mathbf{An}}$ be the category of (complex) analytic spaces. Given an analytic space X, let $|X|$ denote its underlying topological space. A morphism of analytic spaces is etale if it is an isomorphism locally in the analytic topology. Covering families for the (big) etale site $\underline{\mathbf{An}}_{ét}$ (sorry for the notational inconsistency compared to $S_{ét}$ and $(\underline{\mathbf{Sch}}/S)_{ét}$) are given by jointly surjective families of etale morphisms.

An analytic space X gives rise to a stack over $\underline{\mathbf{An}}_{ét}$ via its functor of points. We will not distinguish between an analytic space and its associated stack. A stack $\mathcal{Y}$ over $\underline{\mathbf{An}}_{ét}$ is called an analytic stack, and is representable if it is isomorphic to an analytic space. A 1-morphism $\mathcal{U} \to \mathcal{V}$ of analytic stacks is representable if for any analytic space X and any 1-morphism $X \to \mathcal{V}$, the 2-fiber product $\mathcal{U} \times_{\mathcal{V}} X$ is representable. If P is a property of morphisms in $\underline{\mathbf{An}}$ that is stable under base change (e.g. etale, surjective, separated, flat, proper), then a representable 1-morphism of analytic stacks $\mathcal{U} \to \mathcal{V}$ has P if for any analytic space X and any 1-morphism $X \to \mathcal{V}$, the morphism of analytic spaces $\mathcal{U} \times_{\mathcal{V}} X \to X$ has P.

3 Deligne-Mumford stacks

From now on, let $\mathcal{C}$ be the site $\underline{\mathbf{An}}_{ét}$ or a full subcategory of $(\underline{\mathbf{Sch}}/S)_{ét}$ (with the same notion of coverings) for some fixed noetherian base scheme S. A Deligne-Mumford stack over $\mathcal{C}$ is a stack in groupoids $p : \mathcal{X} \to \mathcal{C}$ such that:

(a) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by schemes).

(b) There is an object $U \in \mathcal{C}$ with a surjective etale morphism $f : U \to \mathcal{X}$[5].

[3] More precisely (see [Sta16] 06VE), the inclusion functor is fully faithful, right exact, compatible with colimits and tensor products...

[4] this is the same as the kernel viewed sheaves of abelian groups, since the forgetful functor $\underline{\mathbf{Mod}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét}, \mathcal{O}) \to \underline{\mathbf{Ab}}((\underline{\mathbf{Sch}}/\mathbb{Z})_{ét})$ is exact [Sta16] 03DA. In particular such kernels can be readily detected on sections.

[5] Note that this implies that Δ must be unramified [Sta16] 06MB

In addition, to fit with the assumptions of [Hal14] (which we use as our reference for GAGA), we will assume that:

(c) $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasicompact and separated.

(d) We may find U as above such that the composition $U \to \mathcal{X} \to S$ is locally of finite type.

Note that for objects $U, V \in \mathcal{C}$ and a morphism $x : U \to \mathcal{X}$ and $y : V \to \mathcal{X}$, we have a cartesian diagram

$$\begin{array}{ccc} U \times_{x,\mathcal{X},y} V & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta_{\mathcal{X}} \\ U \times V & \xrightarrow{(x,y)} & \mathcal{X} \times \mathcal{X} \end{array}$$

which shows that any morphism from an analytic space to a stack with representable diagonal is itself representable. In particular, (b) makes sense in light of (a). A Deligne-Mumford stack over $\underline{\mathbf{An}}_{\acute{e}t}$ is called an analytic DM stack. A DM stack over $(\underline{\mathbf{Sch}}/S)_{\acute{e}t}$ is called an algebraic DM stack.

3.1 Quasicoherent sheaves on Deligne Mumford stacks

As usual, when speaking of sheaves on stacks, a sheaf on $\mathcal{X}$ will refer to a sheaf on the site $\mathcal{X}$ equipped with the topology inherited from $\mathcal{C}$.

Our references to stacky GAGA consider only quasicoherent sheaves on the small etale sites of Deligne Mumford stacks (see below §3.2), whereas the discussion above only treats quasicoherent sheaves on big sites. However, for Deligne-Mumford stacks, because such stacks have a presentation describable in the small étale site, the resulting notions of quasicoherent sheaves are equivalent. That is to say, on a DM stack, a quasicoherent sheaf on the small site uniquely extends to a quasicoherent sheaf on the big site. The precise details/references are as follows.

Let $p : \mathcal{X} \to \mathcal{C}$ be a Deligne-Mumford stack, where $\mathcal{C}$ is either $\underline{\mathbf{An}}_{\acute{e}t}$ or $(\underline{\mathbf{Sch}}/S)_{\acute{e}t}$ for some fixed base scheme S with the natural structure sheaf. Let $U \to \mathcal{X}$ be a etale cover with $U \in \mathcal{C}$. Then, $R := U \times_{\mathcal{X}} U \in \mathcal{C}$ admits two maps $s, t : R \rightrightarrows U$, as well as a map $c : R \times_{s,U,t} R \to R$ given by

$$c : R \times_{s,U,t} R = U \times_{\mathcal{X}} U \times_{\mathcal{X}} U \stackrel{\mathrm{pr}_{02}}{\longrightarrow} R$$

The data (U, R, s, t, c) is a groupoid in $\mathcal{C}$ ([Sta16] 0230), and the natural map $U \to \mathcal{X}$ induces an equivalence of categories $[U/R] \cong \mathcal{X}$ ([Sta16] 04T5).

By definition, a quasi-coherent module on (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module (ie, a module on U_{Zar}), and α is an $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$$

satisfying a certain cocycle condition which is difficult to draw (see [Sta16] 03LI).

In the above, it may seem more correct to require that $\mathcal{F}$ be a quasicoherent module on $U_{\acute{e}t}$ instead of U_{Zar}. However, it turns out the categories of quasicoherent sheaves on these sites are equivalent ([Sta16] 03DR)[6]

Proposition 3.1.1. *The category of quasicoherent modules on $\mathcal{X}$ is equivalent to the category of quasicoherent modules on (U, R, s, t, c).*

Proof. Roughly speaking, given a quasicoherent module on $\mathcal{X} \cong [U/R]$, one can restrict it to obtain a quasicoherent module on U, and one can show that this satisfies the appropriate compatibilities for it to define a quasicoherent module on (U, R, s, t, c). For the other direction, a key point is that given a category fibered in groupoids over a ringed site, the stackification map induces an equivalence on the categories of sheaves, sheaves of modules, and quasicoherent sheaves of modules ([Sta16] 06WP). Thus, given a quasicoherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c), it suffices to construct a quasicoherent module on the prestack quotient $[U/_pR]$. To do this, for any morphism $t : T \to U$, one simply defines $\mathcal{F}(T, t) := \Gamma(T, t^*\mathcal{F})$. See [Sta16] 06WT for details. □

[6] It seems that one needs to be somewhat careful when speaking about properties of quasicoherent sheaves viewed on $U_{\acute{e}t}$ vs U_{Zar}. The only cases one needs to be careful are for the properties of locally free and coherent (though finite locally free is fine), and for coherent, there is no difference as long as U is locally noetherian, which will always be the case in the following.

3.2 Sheaves on the small site $\mathcal{X}_{ét}$

Given a DM stack $p : \mathcal{X} \to \mathcal{C}$, let $\mathcal{X}_{ét}$ denote the *small etale site* of $\mathcal{X}$[7]. That is, its objects are etale morphisms $U \to \mathcal{X}$ with $U \in \mathcal{C}$, and coverings are jointly surjective families of etale morphisms. Its structure sheaf, which we will also denote by $\mathcal{O}_\mathcal{X}$ is just the restriction of the usual structure sheaf to $\mathcal{X}_{ét}$. Our references for GAGA ([Hal14], [Toe99]) will consider quasicoherent sheaves on $(\mathcal{X}_{ét}, \mathcal{O}_\mathcal{X})$. By 3.1.1 (also see [Sta16] 06WK), any such quasicoherent sheaf extends uniquely to a quasicoherent sheaf on $\mathcal{X}$, and it's clear that the notions of finite locally free and coherent of §2.4 agree whether we are speaking about sheaves on $\mathcal{X}$ or $\mathcal{X}_{ét}$.

We will let $\underline{\mathbf{QCoh}}(\mathcal{X})$ (resp. $\underline{\mathbf{Coh}}(\mathcal{X})$) be the categories of quasicoherent (resp. coherent) $\mathcal{O}_\mathcal{X}$ modules on $\mathcal{X}$. Let $\underline{\mathbf{Mod}}(\mathcal{X}_{ét})$ denote the category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{ét}$.

3.3 Global sections of sheaves

If $\mathcal{X}$ is DM stack and $\mathcal{F}$ is a quasicoherent sheaf on $\mathcal{X}$, a priori we may only evaluate $\mathcal{F}$ on objects of the category $\mathcal{X}$ or $\mathcal{X}_{ét}$, which may not have a final object. Nonetheless, we may define its global sections as

$$\mathcal{F}(\mathcal{X}) := \Gamma(\mathcal{X}, \mathcal{F}) := H^0(\mathcal{X}, \mathcal{F}) := \mathrm{Hom}_{\underline{\mathbf{PSh}}(\mathcal{X})}(e, \mathcal{F}) = \mathrm{Hom}_{\underline{\mathbf{Mod}}(\mathcal{X}_{ét})}(\mathcal{O}_\mathcal{X}, \mathcal{F}) \qquad \text{(c.f. [Sta16] 071D)}$$

where e is the final object in the category of presheaves of sets on $\mathcal{X}$[8]. By default, this set of global sections only has the structure of an abelian group, though if $\mathcal{C}$ (with structure sheaf $\mathcal{O}$) has a final object t, then $\mathcal{O}(t)$ is a ring, and $H^0(\mathcal{X}, \mathcal{F})$ has the structure of an $\mathcal{O}(t)$-module. By definition, a global section of $\mathcal{F}$ is thus the data of a section of $\mathcal{F}(x)$ for every $x \in \mathcal{X}$, compatible with all morphisms of the category $\mathcal{X}$. If $\mathcal{X}$ is representable by a scheme/analytic space X then X is a final object of $\mathcal{X}, \mathcal{X}_{ét}$, and so we may take global sections by evaluating $\mathcal{F}$ on X.

Example 3.3.1. While this definition obviously agrees with the classical definition for global sections of sheaves on schemes, the stacky nature of $\mathcal{X}$ gives the definition an added subtlety. For example, a homomorphism $\sigma : \mathcal{O}_\mathcal{X} \to \mathcal{F}$ is a natural transformation of functors, and hence such homomorphisms must be compatible with the morphisms in the category $\mathcal{X}$. In particular, for $T \in \mathcal{C}$ and an automorphism $a : x \to x$ in the fiber category $\mathcal{X}(T)$[9], we have automorphisms $\mathcal{O}_\mathcal{X}(a) : \mathcal{O}_\mathcal{X}(x) \to \mathcal{O}_\mathcal{X}(x)$ and $\mathcal{F}(a) : \mathcal{F}(x) \to \mathcal{F}(x)$ (in the "opposite direction"). By the definition of the structure sheaf, the fact that a lies over id_T means that $\mathcal{O}_\mathcal{X}(a) = \mathrm{id}_{\mathcal{O}_\mathcal{X}(x)}$, but $\mathcal{F}(a)$ may still be nontrivial. Thus, the functoriality of σ says that the following diagram must commute:

$$\begin{array}{ccc} \mathcal{O}_\mathcal{X}(x) & \xrightarrow{\sigma(x)} & \mathcal{F}(x) \\ {\scriptstyle \mathcal{O}_\mathcal{X}(a) = \mathrm{id}_{\mathcal{O}_\mathcal{X}(x)}} \downarrow & & \downarrow {\scriptstyle \mathcal{F}(a)} \\ \mathcal{O}_\mathcal{X}(x) & \xrightarrow{\sigma(x)} & \mathcal{F}(x) \end{array}$$

This says exactly that the image of $1 \in \mathcal{O}_\mathcal{X}(x)$ under $\sigma(a)$ should be invariant under $\mathcal{F}(\mathrm{Aut}_{\mathcal{X}(T)}(x))$. This condition corresponds precisely to the transformation law satisfied by modular forms (c.f. Definition 6.2.1 below).

Alternatively, using the equivalence of categories 3.1.1, we may also take global sections via:

$$\mathcal{F}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{F}) = \mathrm{Eq}\left(\mathcal{F}(U) \overset{s^*, t^*}{\rightrightarrows} \mathcal{F}(U \times_\mathcal{X} U)\right) = \mathrm{Ker}\left(\mathcal{F}(U) \xrightarrow{s^* - t^*} \mathcal{F}(U \times_\mathcal{X} U)\right)$$

4 GAGA

Let $\underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C}$ denote the category of schemes locally of finite type over $\mathbb{C}$.

[7]this notation agrees with [Hal14], but disagrees with [Sta16] 06TP.

[8]Specifically, e is the constant presheaf with value the singleton set

[9]being a morphism in the fiber category here means that a lies over id_T in $\mathcal{C}$

4.1 GAGA for schemes

This is quoted from SGA [Gro71] Exposé XII, *Geometrie algebrique et geometrie analytique.*

Let $\underline{\mathbf{RS}}_{\mathbb{C}}$ denote the category of spaces ringed in $\mathbb{C}$-algebras. If X is a scheme locally of finite type over $\mathbb{C}$, we may associate to X the functor:

$$\underline{\mathbf{An}} \to \underline{\mathbf{Sets}} \qquad Z \mapsto \mathrm{Hom}_{\underline{\mathbf{RS}}_{\mathbb{C}}}(Z, X)$$

By [Gro71] §XII.1.1, this functor is representable by an analytic space, denoted X^{an}, which is equipped with a canonical morphism $\varphi : X^{\mathrm{an}} \to X$ in $\underline{\mathbf{RS}}_{\mathbb{C}}$ inducing an isomorphism of functors $\underline{\mathbf{An}} \to \underline{\mathbf{Sets}}$:

$$\varphi_* : \mathrm{Hom}_{\underline{\mathbf{RS}}_{\mathbb{C}}}(*, X^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathbf{RS}}_{\mathbb{C}}}(*, X) \qquad f \mapsto \varphi \circ f$$

In particular, the map φ induces a bijection $|X^{\mathrm{an}}| \to X(\mathbb{C})$, and the induced maps on local rings are local homomorphisms which induce isomorphisms on their completions. By the definition of $X^{\mathrm{an}} \to X$, for any morphism $X \to Y$ of locally finite type $\mathbb{C}$-schemes, the map $X^{\mathrm{an}} \to X \to Y$ factors uniquely through Y^{an}, and hence the association $X \mapsto X^{\mathrm{an}}$ defines a functor called *analytification*

$$\mathrm{an} : \underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C} \to \underline{\mathbf{An}} \qquad X \mapsto X^{\mathrm{an}}$$

To any locally finite type scheme $X/\mathbb{C}$ with analytification $\varphi : X^{\mathrm{an}} \to X$ and $\mathcal{O}_X$-module F, we may form the pullback $\varphi^* F$, which is $\mathcal{O}_{X^{\mathrm{an}}}$-module. The association

$$F \mapsto F^{\mathrm{an}} := \varphi^* F$$

gives rise to a functor $\underline{\mathbf{Mod}}(\mathcal{O}_X) \to \underline{\mathbf{Mod}}(\mathcal{O}_{X^{\mathrm{an}}})$ commuting with all inductive limits and takes coherent modules to coherent modules ([Gro71] §XII.1.3). Moreover, this functor is exact and faithful (hence conservative).

If X is a proper $\mathbb{C}$-scheme, then analytification gives an equivalence of categories $\underline{\mathbf{Coh}}(X) \cong \underline{\mathbf{Coh}}(X^{\mathrm{an}})$ (c.f. [Gro71] §XII.4.4).

4.2 GAGA for stacks

This is mostly quoted from [Hal14] §2 and Toen [Toe99] §5.

There seem to be two notions of analytification for stacks. Let $\mathcal{X}$ be an algebraic Deligne-Mumford stack over $(\underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C})_{\acute{e}t}$, with an etale covering given by $U \to \mathcal{X}$ with a U scheme. Let $R := U \times_{\mathcal{X}} U$. The analytification functor $\mathrm{an} : \underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C} \to \underline{\mathbf{An}}$ is continuous (for the etale topology), and we let

$$\alpha : \underline{\mathbf{An}}_{\acute{e}t} \to (\underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C})_{\acute{e}t}$$

denote the corresponding morphism of (big etale) sites. In the rest of this section we may omit the subscript *ét*, as we will only consider the etale topology on $\underline{\mathbf{An}}$ and $\underline{\mathbf{Sch}}^{\mathrm{LoFT}}/\mathbb{C}$.

We may define the analytification $\mathcal{X}^{\mathrm{an}}$ to be:

- $\mathcal{X}^{\mathrm{an}} := [R^{\mathrm{an}} \rightrightarrows U^{\mathrm{an}}]$ (as in Hall [Hal14] §2) or
- $\mathcal{X}^{\mathrm{an}} := \alpha^* \mathcal{X}$ (as in Toen [Toe99] Lemme 5.5)

  This is denoted $\alpha^{-1}\mathcal{X}$ in [Sta16] Tag 04WJ. Presumably this agrees with Toen's reference pointing to a definition of Giraud [Gir71] §II.3.2.

We will assume that the two definitions agree (I have not checked it myself: presumably we can show that they agree on schemes, and that they "preserve" presentations).

As with schemes, the association $\mathcal{X} \rightsquigarrow \mathcal{X}^{\mathrm{an}}$ is functorial, and from Hall's definition it is clear that we have a bijection of sets of points $|\mathcal{X}^{\mathrm{an}}| = |\mathcal{X}(\mathbb{C})|$.

An algebraic DM stack $\mathcal{X}$ is proper if and only if $\mathcal{X}^{\mathrm{an}}$ is ([Hal14] §2). For any proper algebraic DM stack $\mathcal{Y}$, the functor induced by analytification:

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}(\mathcal{X}^{\mathrm{an}}, \mathcal{Y}^{\mathrm{an}})$$

is an equivalence of categories ([Hal14] Theorem C).

Let $\underline{\mathbf{Coh}}^{\text{alg}} \to \underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$ denote the stack of (algebraic) coherent sheaves. Its objects consist of pairs $(U, \mathcal{F})$ where $U \in \underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$ and $\mathcal{F}$ is a coherent sheaf on U. The structure morphism to $\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$ just forgets the sheaf $\mathcal{F}$. For two objects $(U, \mathcal{F}), (V, \mathcal{G})$, a morphism from $(U, \mathcal{F})$ to $(V, \mathcal{G})$ is a pair (f, b) where $f : U \to V$ is a morphism and $b : f^*\mathcal{G} \to \mathcal{F}$ is an isomorphism in $\underline{\mathbf{Coh}}(U)$. We have an analogous definition of the stack $\underline{\mathbf{Coh}}^{\text{an}} \to \underline{\mathbf{An}}$ of coherent analytic sheaves. Clearly both $\underline{\mathbf{Coh}}^{\text{alg}}, \underline{\mathbf{Coh}}^{\text{an}}$ are stacks in groupoids.

We may form the pushforward stack $\alpha_*\underline{\mathbf{Coh}}^{\text{an}}$, which is a stack in groupoids over $(\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C})_{\acute{e}t}$. By definition (c.f. [Sta16] Tag 04WA) the objects of $\alpha_*\underline{\mathbf{Coh}}^{\text{an}}$ are pairs $(U, \mathcal{F})$ where $U \in \underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$ and $\mathcal{F} \in \underline{\mathbf{Coh}}(U^{\text{an}})$. The morphisms of $\alpha_*\underline{\mathbf{Coh}}^{\text{an}}$ are pairs $(a, b) : (U, \mathcal{F}) \to (V, \mathcal{G})$ where $a : U \to V$ is a morphism in $\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$ and $b : (a^{\text{an}})^*\mathcal{G} \to \mathcal{F}$ is an isomorphism in $\underline{\mathbf{Coh}}(U^{\text{an}})$.

We may also form a pullback stack $\alpha^*\underline{\mathbf{Coh}}^{\text{alg}}$, which is a stack in groupoids over $\underline{\mathbf{An}}_{\acute{e}t}$. The definition is somewhat complicated (c.f. [Sta16] Tag 04WA), but the key point is that α^* is left adjoint to α_*, and this gives a canonical equivalence of categories

$$\text{Mor}_{\underline{\mathbf{Stacks}}/\underline{\mathbf{An}}}(\alpha^*\underline{\mathbf{Coh}}^{\text{alg}}, \underline{\mathbf{Coh}}^{\text{an}}) \cong \text{Mor}_{\underline{\mathbf{Stacks}}/\mathbb{C}}(\underline{\mathbf{Coh}}^{\text{alg}}, \alpha_*\underline{\mathbf{Coh}}^{\text{an}}) \qquad \text{(c.f. [Sta16] Tag 04WK)}$$

where $\underline{\mathbf{Stacks}}/\mathbb{C}$ refers to the (2,1)-category of stacks in groupoids over $(\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C})_{\acute{e}t}$.

For $X \in (\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C})$, the analytification functor $\underline{\mathbf{Coh}}(X) \to \underline{\mathbf{Coh}}(X^{\text{an}})$ defines a morphism in $\underline{\mathbf{Stacks}}/\mathbb{C}$

$$\underline{\mathbf{Coh}}^{\text{alg}} \to \alpha_*\underline{\mathbf{Coh}}^{\text{an}} \quad \text{defined by} \quad (U, \mathcal{F}) \mapsto (U, \mathcal{F}^{\text{an}})$$

and hence by adjointness, we obtain a morphism in $\underline{\mathbf{Stacks}}/\underline{\mathbf{An}}$

$$\psi : \alpha^*\underline{\mathbf{Coh}}^{\text{alg}} \to \underline{\mathbf{Coh}}^{\text{an}}$$

For an algebraic DM stack $\mathcal{X}$ over $\underline{\mathbf{Sch}}^{\text{LoFT}}/\mathbb{C}$, we have equivalences of categories

$$\begin{aligned} \underline{\mathbf{Coh}}(\mathcal{X}) &\cong \text{Hom}_{\underline{\mathbf{Stacks}}/\mathbb{C}}(\mathcal{X}, \underline{\mathbf{Coh}}^{\text{alg}}) \\ \underline{\mathbf{Coh}}(\mathcal{X}^{\text{an}}) &\cong \text{Hom}_{\underline{\mathbf{Stacks}}/\underline{\mathbf{An}}}(\mathcal{X}^{\text{an}}, \underline{\mathbf{Coh}}^{\text{an}}) \end{aligned}$$

which essentially come from the definition of a coherent sheaf on a stack. Thus, recalling that $\mathcal{X}^{\text{an}} := \alpha^*\mathcal{X}$, we now have a functor

$$\begin{array}{ccccccc} \underline{\mathbf{Coh}}(\mathcal{X}) \cong & \text{Hom}_{\underline{\mathbf{Stacks}}/\mathbb{C}}(\mathcal{X}, \underline{\mathbf{Coh}}^{\text{alg}}) & \to & \text{Hom}(\mathcal{X}^{\text{an}}, \alpha^*\underline{\mathbf{Coh}}^{\text{alg}}) & \to & \text{Hom}(\mathcal{X}^{\text{an}}, \underline{\mathbf{Coh}}^{\text{an}}) & \cong \underline{\mathbf{Coh}}(\mathcal{X}^{\text{an}}) \\ & \mathcal{F} & \mapsto & \alpha^*\mathcal{F} & \mapsto & (\psi \circ \alpha^*\mathcal{F}) & \end{array}$$

This defines the analytification functor for coherent sheaves on Deligne-Mumford stacks:

$$\text{an} : \underline{\mathbf{Coh}}(\mathcal{X}) \to \underline{\mathbf{Coh}}(\mathcal{X}^{\text{an}})$$

If $\mathcal{X}$ is proper, then this functor is an equivalence of categories (c.f. [Toe99] §5.10, [Hal14] §2.4).

5 Moduli of elliptic curves

5.1 A universal family

Let $\mathcal{M}(1)$ denote the moduli stack of elliptic curves over $\mathbb{C}$, and $\mathcal{M}(1)^{\text{an}}$ the analytic moduli stack of elliptic curves - this is a stack over $\underline{\mathbf{An}}$. Let $\mathcal{E}$ denote the universal (stacky) elliptic curve over $\mathcal{M}(1)$, and $\mathcal{E}^{\text{an}}$ the universal curve over $\mathcal{M}(1)^{\text{an}}$. Given $T \in \underline{\mathbf{An}}$ and a morphism $f : T \to \mathcal{M}(1)^{\text{an}}$, let $f^*\mathcal{E}^{\text{an}}$ denote the elliptic curve over T corresponding to f. In this section we will construct an explicit family of elliptic curves $\mathbb{E}$ over the upper half plane $\mathcal{H}$ such that $\mathcal{E} = [\mathbb{E}/\operatorname{SL}_2(\mathbb{Z})]$.

For any $\tau \in \mathcal{H}$, let $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, and $E_\tau := \mathbb{C}/\Lambda_\tau$. Over $\mathcal{H}$, we have a "universal family" of elliptic curves $\mathbb{E} := (\mathcal{H} \times \mathbb{C})/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts freely by the rule

$$(a, b) \cdot (\tau, z) := (\tau, z + a\tau + b) \qquad a, b \in \mathbb{Z}, \quad \tau \in \mathcal{H}, \quad z \in \mathbb{C}$$

Thus, $\mathbb{E}$ is a family of elliptic curves over $\mathcal{H}$. In this section we will show that this family carries a canonical "framing" which makes it into a universal family of framed elliptic curves.

The natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ lifts to an action on $\mathbb{E}$ defined as follows:

$$\tilde{\gamma}(\tau, z) := \left(\gamma\tau, \frac{1}{cz+d} \cdot z\right) \qquad \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) \tag{2}$$

Thus, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have a pullback diagram

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\tilde{\gamma}} & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \xrightarrow{\gamma} & \mathcal{H} \end{array}$$

It is a crucial fact, which is straightforward to check, that for any $\tau, \tau' \in \mathcal{H}$, we have a bijection

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\tau = \tau'\} \cong \mathrm{Isom}(\mathbb{E}_\tau, \mathbb{E}_{\tau'}) \qquad \gamma \mapsto \tilde{\gamma}|_{\mathbb{E}_\tau} : \tag{3}$$

and moreover, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then $\tilde{\gamma}|_{\mathbb{E}_\tau} : \mathbb{E}_\tau \to \mathbb{E}_{\tau'}$ induces the multiplication-by-$j(\gamma, \tau) := \frac{1}{c\tau+d}$ map on tangent spaces at the origin. In particular, $\mathrm{Aut}(\mathbb{E}_\tau) \cong \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\tau)$, and the map $j(*, \tau)$ gives an isomorphism

$$\begin{array}{rcl} j(*, \tau) : \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\tau) & \xrightarrow{\sim} & \mu_n := \{e^{2\pi i/k} : k \in \{0, 1, 2, \dots, n-1\}\} \\ \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] & \mapsto & j(\gamma, \tau) := \dfrac{1}{c\tau + d} \end{array}$$

For any fixed τ, τ', the set $\mathrm{Isom}(\mathbb{E}_\tau, \mathbb{E}'_\tau)$ is a torsor under $\mathrm{Aut}(\mathbb{E}_\tau) \cong \mu_n$, and the set of $j(\gamma, \tau)$ for γ sending $\tau \mapsto \tau'$ is also a torsor under μ_n. In particular, the values $j(\gamma, \tau)$ are distinct as γ ranges over $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\tau)$.

A *framing* (c.f. [Hai08] Definition 1.13) on an elliptic curve E is an ordered basis v_1, v_2 of $H_1(E, \mathbb{Z})$ such that the intersection number $v_1 \cdot v_2 = 1$. By Ehresmann's fibration theorem, any family of elliptic curves E/T is a locally trivial C^∞ fiber bundle. For a contractible open $U \subset T$ and $s, t \in U$, the inclusions of E_s, E_t into E_U are homotopy equivalences, and hence induce isomorphisms

$$H_1(E_s, \mathbb{Z}) \cong H_1(E_U, \mathbb{Z}) \cong H_1(E_t, \mathbb{Z}) \tag{4}$$

A locally constant framing on E/T is a family

$$\{v_1(t), v_2(t) \in H_1(E_t, \mathbb{Z}) : v_1(t) \cdot v_2(t) = 1,\ t \in T\}$$

such that for every contractible open $U \subset T$ and $s, t \in U$, $(v_1(s), v_2(s))$ maps to $(v_1(t), v_2(t))$ under the isomorphism (4). A family of elliptic curves E/T is *framed* if it is equipped with a locally constant framing.

The family $\mathbb{E}$ is equipped with a universal locally constant framing, where over $\tau \in \mathcal{H}$, $\mathbb{E}_\tau$ is framed by the homology classes represented by the "straight" path going from $0 \rightsquigarrow 1 \in \mathbb{C}/\Lambda_\tau$, and $0 \rightsquigarrow \tau \in \mathbb{C}/\Lambda_\tau$. A framed family of elliptic curves $(E/T, v_1, v_2)$ determines a period mapping

$$\Phi_{E/T} : T \to \mathcal{H} \qquad t \mapsto \frac{\int_{v_1(t)} \omega_t}{\int_{v_2(t)} \omega_t}$$

where for every t we choose some nonzero differential ω_t on E_t. It's clear from the formula that the period map $\Phi_{E/T}$ does not depend on this choice of ω_t. This period map is holomorphic ([Hai08] Proposition 2.3) and moreover induces a unique isomorphism $\mathbb{E}_T \cong E$ preserving the framings. In particular, $\mathcal{H}$ is a fine moduli space for framed families of elliptic curves ([Hai08], Proposition 2.4).

Proposition 5.1.1. *Let $\mathcal{M}(1)^{an}$ denote the analytic moduli stack of elliptic curves over $\underline{\boldsymbol{An}}_{\acute{e}t}$. The family $\mathbb{E}/\mathcal{H}$ determines a map $\mathcal{H} \to \mathcal{M}(1)^{an}$ which factors, via the canonical covering map $\mathrm{pr}_{\mathrm{SL}_2(\mathbb{Z})} : \mathcal{H} \to [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})]$, through an equivalence of stacks $[\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})] \cong \mathcal{M}(1)^{an}$.*

Proof. Let $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})]$ be the category fibered in groupoids over $\underline{\mathbf{An}}$ defined as follows[10].

- Given an object $T \in \underline{\mathbf{An}}$, the fiber category $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})](T)$ is the category whose objects are morphisms $T \to \mathcal{H}$, and given $a, b : T \to \mathcal{H}$, $\mathrm{Mor}_{[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})](T)}(a, b) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \circ a = b\}$. We will write such a morphism as a triple (γ, a, b), or as $\gamma : a \to b$.
- The objects of $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})]$ are morphisms $T \to \mathcal{H}$, and its image in $\underline{\mathbf{An}}$ is T.
- Given two objects $a : T \to \mathcal{H}$ and $a' : T' \to \mathcal{H}$, the set of morphisms $\mathrm{Mor}_{[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})]}(a, a')$ is the disjoint union
$$\bigsqcup_{f \in \mathrm{Mor}_{\underline{\mathbf{An}}}(T, T')} \mathrm{Mor}_{[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})](T)}(a, f^*a')$$
where $f^*a' := a \circ f$. The structure morphism $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})] \to \underline{\mathbf{An}}$ is given by sending $a : T \to \mathcal{H}$ to T, and sending a morphism in the above disjoint union to the corresponding f.

Then, by definition, there is a morphism $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})] \to [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})]$ identifying the latter as the stackification of the former. We will construct a functor

$$F : [\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})] \to \mathcal{M}(1)^{\mathrm{an}}$$

as follows. To any object $a : T \to \mathcal{H}$ in $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})](T)$, let $F(a) := a^*\mathbb{E}$, viewed as an elliptic curve over T. For two objects $a, b : T \to \mathcal{H}$ in $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})]$, if $\gamma : a \to b$ in the fiber category over T, then $b = \gamma \circ a$, and hence $\tilde{\gamma}$ determines a unique isomorphism $a^*\mathbb{E} \xrightarrow{\sim} b^*\mathbb{E}$ making the natural diagram commute. We define $F(\gamma : a \to b)$ to be this isomorphism. This in turn determines the functor F on all morphisms in $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})]$.

It's clear from definition that the composition

$$\mathcal{H} \xrightarrow{\mathrm{pr}_{\mathrm{SL}_2(\mathbb{Z})}} [\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})] \xrightarrow{F} \mathcal{M}(1)^{\mathrm{an}}$$

is the morphism determined by the family $\mathbb{E}/\mathcal{H}$. Thus, it remains to show that the functor $F : [\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})] \to \mathcal{M}(1)^{\mathrm{an}}$ is both a monomorphism and an epimorphism. If this is the case, then by [Noo05] §3.5, it would follow that F induces an equivalence on stackifications $[\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})] \xrightarrow{\sim} \mathcal{M}(1)^{\mathrm{an}}$.

By definition ([Noo05] §3.1), to show that F is a monomorphism, one must show that the restriction of F to fiber categories $[\mathcal{H}/_p \mathrm{SL}_2(\mathbb{Z})](T)$ is fully faithful. Faithfulness is clear from the construction. To show fullness, we wish to show that for any $a, b : T \to \mathcal{H}$ and any isomorphism $\sigma : a^*\mathbb{E} \xrightarrow{\sim} b^*\mathbb{E}$, there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ a = b$ and the following diagram commutes:

$$\begin{array}{ccc} b^*\mathbb{E} & \longrightarrow & \mathbb{E} \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\tilde{\gamma}} \\ a^*\mathbb{E} & \longrightarrow & \mathbb{E} \end{array} \tag{5}$$

We can reduce to the case where T is connected. The relative tangent bundle at the zero section of $\mathbb{E}/\mathcal{H}$ is visibly trivial, and we may identify it with $\mathcal{H} \times \mathbb{C}$. Being pullbacks of $\mathbb{E}$, the relative tangent bundles at the zero sections of $a^*\mathbb{E}, b^*\mathbb{E}$ are also trivial, and we will use a, b to identify them with $T \times \mathbb{C}$. The isomorphism σ induces an isomorphism on relative tangent bundles $d\sigma : T \times \mathbb{C} \to T \times \mathbb{C}$ over T. For any $t \in T$, $\sigma_t := \sigma|_{(a^*\mathbb{E})_t}$ can be viewed as an isomorphism

$$\sigma_t : \mathbb{E}_{a(t)} \to \mathbb{E}_{b(t)}$$

which by (3), is precisely $\tilde{\gamma_t}|_{\mathbb{E}_{a(t)}}$ for some $\gamma_t \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $(\gamma_t \circ a)(t) = b(t)$. We claim that $\tilde{\gamma_t}$ makes (5) commute. To see this, for every $t \in T$, the morphism $d\sigma_t : t \times \mathbb{C} \to t \times \mathbb{C}$ is given by multiplication by some complex number, which by the above discussion must be precisely $j(\gamma_t, a(t)) := \frac{1}{c_t a(t) + d_t}$, where $\gamma_t := \left[\begin{smallmatrix} a_t & b_t \\ c_t & d_t \end{smallmatrix}\right]$. Thus, $d\sigma$ defines a continuous function $T \to \mathbb{C}$ sending $t \mapsto j(\gamma_t)$. It's values are constrained by the requirement:

$$j(\gamma_t, a(t)) = \frac{1}{c_t a(t) + d_t} \qquad \text{for some } \gamma_t = \left[\begin{smallmatrix} a_t & b_t \\ c_t & d_t \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}), (\gamma_t \circ a)(t) = b(t)$$

[10]This is the prestack quotient of $\mathcal{H}$ by $\mathrm{SL}_2(\mathbb{Z})$, see [Sta16] Tag 044O

For any t, the set of γ_t satisfying $(\gamma_t \circ a)(t) = b(t)$ is finite and give rise to distinct values of $j(\gamma_t, a(t))$, which hence form a discrete subset in $\mathbb{C}$. Thus, since $j(\gamma_t, a(t))$ is continuous in t, it must be the case that $\gamma_t : T \to \mathrm{SL}_2(\mathbb{Z})$ is constant, with value $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In particular, the associated $\tilde{\gamma}$ makes (5) commute, as desired. This completes the proof that the functor F is fully faithful on fiber categories, and hence F is a monomorphism.

To see that it is an epimorphism, we must show that for every elliptic curve E over $U \in \underline{\mathbf{An}}$, there exists a covering $\{p_i : U_i \to U\}$ with each p_i etale (a local isomorphism), such that E_{U_i} is isomorphic to $a_i^*\mathbb{E}$ for some $a_i : U_i \to \mathcal{H}$. To do this, we may cover U with contractible opens U_i, so that each E_{U_i} admits a framing. Choosing such a framing for each U_i, we obtain period maps $a_i : U_i \to \mathcal{H}$ and isomorphisms $E_{U_i} \cong a_i^*\mathbb{E}$, as desired.

□

Definition 5.1.2. For any subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ (not necessarily finite index), the stacky quotient $[\mathcal{H}/\Gamma]$ carries the universal family $\mathbb{E}_\Gamma := [\mathbb{E}/\Gamma]$ (the action being as given in (2)), which we call the universal elliptic curve over $[\mathcal{H}/\Gamma]$.

5.2 The Tate curve

Our main reference for this section is [Sil94] §V.1.

5.2.1 The Tate curve analytically

For $n \geq 1$, let P_n be the cyclic subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrix $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$. Then P_n acts on both $\mathcal{H}$ and $\mathbb{E}$ without fixed points, and hence $\mathbb{E}_{P_n} := [\mathbb{E}/P_n] = \mathbb{E}/P_n$ and $[\mathcal{H}/P_n] = \mathcal{H}/P_n$. The function $q := e^{2\pi i\tau}$ induces a biholomorphism $\mathcal{H}/P_1 \cong D^\circ := \{t \in \mathbb{C}^\times : |t| < 1\}$, and hence $\mathbb{E}/P_1$ defines an elliptic curve over the punctured unit disk D° with parameter q, which we call the (analytic) Tate curve $\mathrm{Tate}^{\mathrm{an}}$. Similarly, for general $n \geq 1$, we have a pullback diagram

$$\begin{array}{ccc} \mathbb{E}_{P_n} & \longrightarrow & \mathbb{E}_{P_1} = \mathrm{Tate}^{\mathrm{an}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}/P_n & \longrightarrow & \mathcal{H}/P_1 \end{array}$$

which identifies $\mathbb{E}/P_n$ with the pullback of $\mathrm{Tate}^{\mathrm{an}}$ by the cyclic n-cover $D^\circ \to D^\circ$. Thus $\mathbb{E}/P_n$ is an elliptic curve over D° with local parameter $q^{1/n}$, which we call the n-sided (analytic) Tate curve $\mathrm{Tate}_n^{\mathrm{an}}$.

Let $\mathcal{H}_{[0,1]} := \{\tau \in \mathcal{H} : \Re(\tau) \in [0,1]\}$, then $\mathrm{Tate}^{\mathrm{an}}$ is also obtained by gluing the two sides of the family $\mathbb{E}|_{\mathcal{H}_{[0,1]}}$ above $\Re(\tau) = 0$ and $\Re(\tau) = 1$ via the "identity map"

$$\mathbb{E}_\tau := \mathbb{C}/\langle 1, \tau\rangle \xrightarrow{\mathrm{id}} \mathbb{C}/\langle 1, \tau+1\rangle =: \mathbb{E}_{\tau+1}$$

Pick some base point $t_0 \in D^\circ$, and let γ denote a "counterclockwise" generator of $\pi_1^{\mathrm{top}}(D^\circ, t_0)$. Then, relative to the canonical framing on $\mathbb{E}$, from the description above it is clear that the monodromy action of γ on $H_1(\mathrm{Tate}_{t_0}^{\mathrm{an}}, \mathbb{Z})$ is given by the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Remark 5.2.1. Let $P_n^* := \langle \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] \rangle \subset \mathrm{SL}_2(\mathbb{Z})$, then note that while the action of P_n and P_n^* on $\mathcal{H}$ are identical, their actions on $\mathbb{E}$ differ by $[-1]$. As a result, $\mathbb{E}/P_n$ and $\mathbb{E}/P_n^*$ give elliptic curves over D°, which are fiberwise identical, but globally nonisomorphic. We call $\mathbb{E}/P_n^*$ the twist of the analytic Tate curve $\mathrm{Tate}^{\mathrm{an}}$. The fact that D° has a unique double cover implies that this is the only nontrivial twist.

Proposition 5.2.2. *Let $V'^\circ \subset D^\circ$ be a small punctured disk. Given a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ and a map $\beta : V'^\circ \to [\mathcal{H}/\Gamma]$ which is "centered at $i\infty$", if $\beta^*\mathbb{E}_\Gamma \cong \mathrm{Tate}_n^{an}|_{V'^\circ}$, then $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$.*

Proof. Here, "centered at $i\infty$" means that a sequence in D° converging to $0 \in D \supset D^\circ$ maps to a sequence in the coarse space $\mathcal{H}/\Gamma$ converging to the cusp $i\infty$ of $\overline{\mathcal{H}/\Gamma}$. Let $\mathrm{cyc}_n : V'^\circ \to D'^\circ$ denote the cyclic n-cover of a small punctured disk $D'^\circ \subset D^\circ$, and let $\beta_1 : D'^\circ \to [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})]$ be the map given by $\mathrm{Tate}^{\mathrm{an}}|_{D'^\circ}$. The composition $\beta_1 \circ \mathrm{cyc}_n : V'^{\circ} \to [\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})]$ is then given by $\mathrm{Tate}_n^{\mathrm{an}}|_{V'^{\circ}} \cong \beta^* \mathbb{E}_\Gamma = \beta^* \mathrm{pr}^* \mathbb{E}_{\operatorname{SL}_2(\mathbb{Z})}$. Thus, we have a (2-)commutative diagram

$$\begin{array}{ccccc} & & & & [\mathcal{H}/\Gamma] \\ & & \overset{\beta}{\nearrow} & & \downarrow{\scriptstyle \mathrm{pr}} \\ V'^{\circ} & \xrightarrow[\mathrm{cyc}_n]{} & D'^{\circ} & \xrightarrow[\mathrm{Tate}^{\mathrm{an}}]{} & [\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})] \end{array}$$

Because β is centered at $i\infty$, from the above discussion, the image of $\pi_1(D'^{\circ})$ inside $\pi_1([\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})]) = \operatorname{SL}_2(\mathbb{Z})$[11] is the cyclic subgroup P_1, and hence the image of $\pi_1(V'^{\circ})$ inside $\pi_1([\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})])$ must be P_n. Since the image of $\pi_1([\mathcal{H}/\Gamma]) \to \pi_1([\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})])$ is Γ, the commutativity of the diagram then implies that $\Gamma \supset P_n$, as desired.

□

5.2.2 The Tate curve algebraically

We recall that $\mathbb{E}_\tau = \mathbb{C}/\langle 1, \tau\rangle$ is described as an algebraic curve in $\mathbb{P}^2_{\mathbb{C}}$ by the equations

$$Y^2 = 4X^3 - g_2(\tau)X - g_3(\tau)$$

where

$$\begin{aligned} q &:= e^{2\pi i \tau} \\ s_k(q) &:= \sum_{n\geq 1} \sigma_k(n) q^n = \sum_{n \geq 1} \frac{n^k q^n}{1-q^n} \\ g_2(\tau) &:= \frac{(2\pi i)^4}{12}(1 + 240 s_3(q)) \\ g_3(\tau) &:= \frac{(2\pi i)^6}{216}(-1 + 504 s_5(q)) \end{aligned}$$

and where the coordinate functions X, Y are given by the Weierstrass function and its derivative $\wp, \wp'$. Since $\wp' := \frac{d\wp}{dz}$, we see that the holomorphic differential dz on $\mathbb{E}_\tau$ corresponds to $\frac{d\wp}{\wp'} = \frac{dX}{Y}$ on the algebraic curve. Sometimes it is useful to make the change of variables

$$X = (2\pi i)^2\left(x + \frac{1}{12}\right) \qquad Y = (2\pi i)^3(2y + x)$$

via which our equation for $\mathbb{E}_\tau$ with differential $2\pi i dz$ becomes the *Tate curve*

$$\mathrm{Tate}(q) : y^2 + xy = x^3 + a_4(q)x + a_6(q) \qquad \text{with differential } \omega_{\mathrm{can}} := \frac{dx}{2y+x} = 2\pi i \frac{dX}{Y} = 2\pi i dz \tag{6}$$

where $a_4(q), a_6(q) \in \mathbb{Z}[\![q]\!]$ are given by

$$a_4(q) = -5 s_3(q) \qquad \text{and} \qquad a_6(q) = -\frac{5 s_3(q) + 7 s_5(q)}{12}$$

One calculates that the discriminant and j-invariant of $\mathrm{Tate}(q)$ is given by:

$$\Delta(\mathrm{Tate}(q)) = \Delta(q) = q \prod_{n \geq 1}(1 - q^n)^{24} \in \mathbb{Z}[\![q]\!]$$

$$j(\mathrm{Tate}(q)) = j(q) = \frac{1}{q} + 744 + 196884q + \cdots \in \mathbb{Z}(\!(q)\!)$$

Thus, by (6) the Tate curve is an elliptic curve over $\mathbb{Z}(\!(q)\!)$. For any $\mathbb{Z}(\!(q)\!)$-algebra R, we define the Tate curve over R to be the base change of $\mathrm{Tate}(q)/\mathbb{Z}(\!(q)\!)$ to R, which we denote $\mathrm{Tate}(q)_R$ over R, or just $\mathrm{Tate}(q)/R$ if no confusion may arise.

Over $\mathbb{C}(\!(q^{1/n})\!)$, the Tate curve has a unique nontrivial twist, and hence is not quite characterized by its j-invariant. It is sometimes useful to distinguish the Tate curve from its twist:

[11] Here, we use the basepoint afforded by the universal cover $\mathcal{H} \to [\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})]$, noting that $\mathcal{H}$ is homotopy-equivalent to a point.

Proposition 5.2.3. *Let E be an elliptic curve over $\mathbb{C}(\!(q^{1/n})\!)$ with j-invariant given by its q-expansion $j(q) = \frac{1}{q} + 744 + \cdots$. Let E_0 be the special fiber of the minimal regular model of E over $\mathbb{C}[\![q^{1/n}]\!]$ at $q^{1/n} = 0$. Viewing $q^{1/n}$ as a uniformizer at $0 \in \mathbb{C}$, let (H) denote the condition*

(H): E admits a Weierstrass equation which defines an analytic family E^{an} of (smooth) elliptic curves over some punctured disk $D^\circ(r)$ of radius $r > 0$ centered at 0. In this case, let γ be a counterclockwise loop generating $\pi_1^{top}(D^\circ(r))$, and let E_1^{an} denote a smooth fiber of E^{an}.

The following are equivalent:

(a) E is the Tate curve over $\mathbb{C}(\!(q^{1/n})\!)$.

(b) E_0 has Kodaira type I_n.

(c) $\operatorname{ord}_{q^{1/n}} \Delta(E) = n$.

(d) If (H) holds, then local monodromy of γ acting on $H_1(E_1^{an}, \mathbb{Z})$ is conjugate to $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Furthermore, the following are equivalent:

(a) E is the (unique) nontrivial twist of the Tate curve over $\mathbb{C}(\!(q^{1/n})\!)$.*

(b) E_0 has Kodaira type I_n^*.*

(c) $\operatorname{ord}_{q^{1/n}} \Delta(E) = 6 + n$.*

(d) If (H) holds, then local monodromy of γ acting on $H_1(E_1^{an}, \mathbb{Z})$ is conjugate to $-\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.*

Proof. Since $\operatorname{ord}_{q^{1/n}}(j(E)) = -n$, the equivalences (b) $\iff$ (c) and (b*) $\iff$ (c*) follows from Tate's algorithm (c.f. Table 4.1 in [Sil94] §IV.9. The equivalences (b) $\iff$ (d) and (b*) $\iff$ (d*) follows from Kodaira's classification of singular fibers (c.f. Table 6 in [BHPVdV04] §V.10). Since $\mathbb{C}(\!(q^{1/n})\!)$ has a unique quadratic extension, $\operatorname{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!)$ has a unique nontrivial twist. Certainly the Tate curve satisfies (c), and it is easy to compute that the unique nontrivial twist of the Tate curve satisfies (c*). This completes the proof. □

Since a_4, a_6 are just linear combinations of $\{1, g_2, g_3\}$, they converge for all $\tau \in \mathcal{H}$, or viewed as functions in q, they converge on the open unit disk $D := \{q \in \mathbb{C} : |q| < 1\}$. Moreover, it is useful to note that at $q = 0$, (6) defines the pointed nodal cubic $\operatorname{Tate}(0) : y^2 + xy = x^3$ with differential $\omega_{\text{can}} := \frac{dx}{2y+x}$, and hence the equation (6) defines a regular and stable (1,1)-curve (resp. analytic family of stable (1,1)-curves) over $\mathbb{Z}[\![q]\!]$ (resp. D), smooth away from the origin. We remark that while $\operatorname{Tate}(q)/\mathbb{Z}[\![q^{1/n}]\!]$ is stable for all $n \geq 1$, it is not regular if $n \neq 1$, and the same is true for its analytification.

Let $f : \operatorname{Tate}(q) \to \operatorname{Spec} \mathbb{C}[\![q^{1/n}]\!]$ be the structure morphism. The direct image dualizing sheaf $f_*\omega_f$ is invertible over $\mathbb{C}[\![q^{1/n}]\!]$ ([DR75] §II, Proposition 1.6). Restriction to $\operatorname{Spec} \mathbb{C}(\!(q^{1/n})\!)$ gives an injection on global sections:

$$H^0(\operatorname{Tate}(q)/\mathbb{C}[\![q^{1/n}]\!], f_*\omega_f) \hookrightarrow H^0(\operatorname{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!), f_*\Omega^1_{\operatorname{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!)})$$

We can calculate that ω_{can} is in the image of this injection:

Proposition 5.2.4. *Let $D^\circ := D - \{0\}$ be the open punctured unit disk. The differential $\omega_{can} := \frac{dx}{2y+x}$ on $\operatorname{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!)$ (resp. $\operatorname{Tate}^{an}|_{D^\circ}$) extends to a basis of the dualizing sheaf on $\operatorname{Tate}(q)/\mathbb{C}[\![q]\!]$ (resp. $\operatorname{Tate}^{an}/D$).*

Proof. We wish to check that $\omega_{\text{can}} = \frac{dx}{2y+x}$ defines a nonzero section of the dualizing sheaf of the nodal cubic $E_0 : y^2 + xy = x^3$. This is equivalent to checking that viewing $\frac{dx}{2y+x}$ as a meromorphic differential, its residues at the preimages of the node of E_0 under the normalization map $\nu : \tilde{E_0} \to E_0$ sum to zero (c.f. [Man99], §V.1.1). We sketch the calculation here. The node of E_0 is situated at $(x, y) = (0, 0)$, and the normalization $\tilde{E_0}$ is isomorphic to $\mathbb{P}^1$ with parameter $t = \frac{y}{x}$, and at the level of functions, the map ν sends $x \mapsto t^2 + t$ and $y \mapsto t(t^2 + t)$. The preimages of the node thus lie at $t = 0, -1$. We have:

$$\omega_{\text{can}} = \frac{dx}{2y + x} = \frac{1}{t} \cdot \frac{2t + 1}{2t^2 + 3t + 1} dt = \frac{1}{t + 1} \cdot \frac{2(t + 1) - 1}{2(t + 1)^2 - 3(t + 1) + 1} d(t + 1)$$

From this it is visible that ω_{can} has residues $1, -1$ at $t = 0, -1$ respectively, which proves the proposition. □

We may also give another characterization of the Tate curve over D°.

Proposition 5.2.5. *The map $D \to \overline{\mathcal{M}(1)}^{an}$ induced by* Tate^{an} *is etale.*

Proof. Let $\overline{M(1)}$ denote the coarse moduli scheme of $\overline{\mathcal{M}(1)}$. The preimage of $i\infty$ under the coarse map $c : \overline{\mathcal{M}(1)} \to \overline{M(1)}$ is represented by the (pointed) nodal cubic Tate(0), and since $\mathrm{Aut}(\mathrm{Tate}(0)) = \mu_2$, all automorphisms of Tate(0) extend to all deformations. Thus, by 8.0.7, the coarse map $c : \overline{\mathcal{M}(1)} \to \overline{M(1)}$ is etale above $i\infty$. The Tate curve over $\mathbb{C}[\![q]\!]$ defines a map $\operatorname{Spec}\mathbb{C}[\![q]\!] \to \overline{\mathcal{M}(1)}$, whose composition

$$\operatorname{Spec}\mathbb{C}[\![q]\!] \to \overline{\mathcal{M}(1)} \to \overline{M(1)}$$

is simply given by taking q-expansion of functions on $\overline{M(1)}$ (as weight 0 modular forms for $\mathrm{SL}_2(\mathbb{Z})$)[12]. Since q is a formal uniformizer at $i\infty \in \overline{M(1)}$, we find that this composition is unramified. Thus, the map $\operatorname{Spec}\mathbb{C}[\![q]\!] \to \overline{\mathcal{M}(1)}$ identifes $\mathbb{C}[\![q]\!]$ with the completion of the etale local ring of $\overline{\mathcal{M}(1)}$ at Tate(0). By Artin approximation (c..f. 8.0.6) this map factors through an etale morphism $U \to \overline{\mathcal{M}(1)}$ with U a finite type $\mathbb{C}$-scheme and $q = 0$ mapping to a point $u \in U$. Taking analytifications and restricting to a suitably small neighborhood of u, we find that the map $D \to \overline{\mathcal{M}(1)}^{\mathrm{an}}$ is etale at $0 \in D$.

To see that $D \to \overline{\mathcal{M}(1)}^{\mathrm{an}}$ is etale at other points, we note that the map $D^\circ \to \overline{\mathcal{M}(1)}^{\mathrm{an}}$ factors as

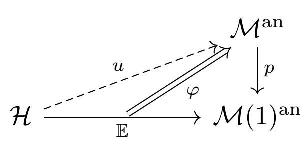
$$D^\circ \cong \mathcal{H}/P_1 \to [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})] \cong \mathcal{M}(1)^{\mathrm{an}} \subset \overline{\mathcal{M}(1)}^{\mathrm{an}},$$

which is obviously etale. □

5.3 Cusps and level structures on the Tate curve

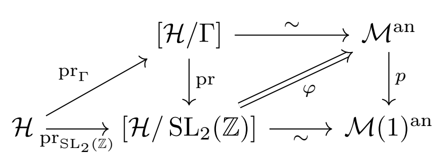
5.3.1 Uniformization of finite etale covers of $\mathcal{M}(1)_{\mathbb{C}}$

In this section, by default everything will be over $\mathbb{C}$. Let $p : \mathcal{M} \to \mathcal{M}(1)$ be a finite[13] étale morphism of connected DM stacks. There is a natural map $\mathcal{H} \to \mathcal{M}(1)^{\mathrm{an}}$ corresponding to the family $\mathbb{E}/\mathcal{H}$ given by sending $\tau \mapsto E_\tau$. Via this map, we will identify $\mathcal{M}(1)^{\mathrm{an}} = [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})]$. Because $\mathcal{H}$ is contractible, by the lifting property of covering maps ([Noo05] §18.18), there exists a lifting $u : \mathcal{H} \to \mathcal{M}^{\mathrm{an}}$ of $\mathcal{H} \xrightarrow{\mathbb{E}} \mathcal{M}(1)^{\mathrm{an}}$ and a 2-isomorphism φ witnessing the 2-commutativity of the following diagram



(7)

Note that the set of all 2-isomorphisms witnessing the (2-)commutativity of the above diagram is a torsor under $\mathrm{Aut}_{\mathcal{H}}(\mathbb{E}) = \{\pm 1\}$. Namely, the possible choices of 2-isomorphisms are $\{\varphi, [-1] \circ \varphi\}$. A choice of such a lifting $\mathcal{H} \to \mathcal{M}^{\mathrm{an}}$ identifies $\mathcal{M}^{\mathrm{an}}$ with $[\mathcal{H}/\Gamma]$ for some finite index $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ such that the diagram (7) uniquely determines a 2-commutative diagram



(8)

where $\mathrm{pr}_\Gamma, \mathrm{pr}_{\mathrm{SL}_2(\mathbb{Z})}, \mathrm{pr}$ denote the canonical projections, and the triangle on the left literally 1-commutes - that is, $\mathrm{pr} \circ \mathrm{pr}_\Gamma = \mathrm{pr}_{\mathrm{SL}_2(\mathbb{Z})}$ "on the nose". More precisely, the triangle 2-commutes, where we may (and will) choose the 2-isomorphism witnessing the commutativity to be the identity. By definition a *uniformization* of $\mathcal{M}^{\mathrm{an}}$ is the data of either the diagram (7) or (8), in particular it includes the 2-isomorphism φ. Given a choice of

[12]This follows from the fact that the function field of $\overline{M(1)}$ is generated by the j-invariant, and the j-invariant of Tate(q) is precisely the q-expansion of the modular function j

[13]We follow the definition of finite as given in [Sta16] 0CHU. In particular, finite implies representable.

uniformization, we may identify $\mathcal{M}^{\mathrm{an}}$ with $[\mathcal{H}/\Gamma]$, so we may sometimes abuse notation and call the isomorphism $\mathcal{M}^{\mathrm{an}} \cong [\mathcal{H}/\Gamma]$ a uniformization of $\mathcal{M}^{\mathrm{an}}$.

If $\mathcal{M}$ has a moduli interpretation, then the choice of a uniformization of $\mathcal{M}^{\mathrm{an}}$ (equivalently, a choice of a 2-commutative diagram as in (7)) can be thought of as an abstract family of $\mathcal{M}$-level structures on $\mathbb{E}/\mathcal{H}$.

5.3.2 Cusps, analytically

The *cusps* of $[\mathcal{H}/\Gamma]$, or just Γ, are the Γ-orbits of points in the boundary $\mathbb{Q} \cup \{\infty\}$ of $\overline{\mathcal{H}}$ (c.f. [DS06] §2.4). The stabilizer of each cusp in Γ is a conjugate of a subgroup of $\pm P = \langle \pm \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \rangle$, and the *width* of the cusp is by definition the minimum positive integer μ such that a conjugate of $\left[\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right]$ lies in Γ and stabilizes the cusp. The *coarse width* of the cusp is by definition the minimum positive integer ν such that a conjugate of $\left[\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right]$ or $-\left[\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right]$ lies in Γ and stabilizes the cusp. If the width equals the coarse width, then the cusp is *regular*. Otherwise, it is *irregular*. Note that for an irregular cusp, the width is always twice the coarse width.

5.3.3 Cusps, framed cusps, and oriented cusps

Suppose $p : \mathcal{M} \to \mathcal{M}(1)$ is finite etale of degree $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. In this section, we will define the notions, in increasing specificity, of "cusp", "framed cusp", and "oriented cusp" of $\mathcal{M}$. Intuitively, if $\mathcal{M}^{\mathrm{an}} = [\mathcal{H}/\Gamma]$, then a cusp of $\mathcal{M}$ is the same data as a usual cusp of the compact Riemann surface $\overline{\mathcal{H}/\Gamma}$. A framed cusp is then the data of a cusp, together with a uniformizer at the corresponding point in the scheme - this corresponds to an isomorphism class of level structures on the Tate curve. Lastly, an oriented cusp is a choice of a particular level structure on the Tate curve. There are 2 oriented cusps lying over any given framed cusp if $-I \notin \Gamma$, and 1 otherwise. In general, given a Katz modular form G, one can only take its q-expansion at an oriented cusp. However, if the form has even weight, or if the cusp is regular, then for any given framed cusp, the q-expansions of G do not depend on the choice of an oriented cusp over a given framed cusp. Thus, for q-expansions, the distinction between framed and oriented is only relevant for odd weight forms at irregular cusps.

Recall that $\mathcal{E}$ is defined to be the universal elliptic curve over the algebraic stack $\mathcal{M}(1)$. Let $\mathcal{E}_{\mathcal{M}} := p^*\mathcal{E}$ be the universal elliptic curve over $\mathcal{M}$. The Tate curve over $\mathbb{C}(\!(q)\!)$ gives a map $\operatorname{Spec}\mathbb{C}(\!(q)\!) \to \mathcal{M}(1)$. The pullback $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)} := \mathcal{M} \times_{\mathcal{M}(1)} \operatorname{Spec}\mathbb{C}(\!(q)\!)$ splits into a disjoint union

$$\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)} \cong \bigsqcup_{i=1}^{r} \operatorname{Spec}\mathbb{C}(\!(q^{1/\mu_i})\!)$$

(a finite etale $\mathbb{C}(\!(q)\!)$-scheme of degree $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$) The points of this scheme $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)}$) are called the *cusps* of the algebraic stack $\mathcal{M}$, and can be identified with $\mathrm{Gal}(\mathbb{C}(\!(q^{1/\infty})\!)/\mathbb{C}(\!(q)\!))$-orbits of morphisms

$$c : \operatorname{Spec}\mathbb{C}(\!(q^{1/\infty})\!) \to \mathcal{M}$$

such that the composition $c \circ p : \operatorname{Spec}\mathbb{C}(\!(q^{1/\infty})\!) \to \mathcal{M}(1)$ is isomorphic to the morphism determined by the Tate curve over $\mathbb{C}(\!(q^{1/\infty})\!)$. The *width* of the cusp of $\mathcal{M}$ given by a point of $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)}$ with residue field $\mathbb{C}(\!(q^{1/\mu_i})\!)$ is defined to be μ_i.

For any $n \geq 1$, a *framed cusp* (valued in $\mathbb{C}(\!(q^{1/n})\!)$) is by definition the (2-)isomorphism class of a morphism $c : \operatorname{Spec}\mathbb{C}(\!(q^{1/n})\!) \to \mathcal{M}$ which factors through $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)}$. The point of $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q)\!)}$ that it factors through is called its "underlying cusp" $|c|$.

For any $n \geq 1$, an *oriented cusp* (valued in $\mathbb{C}(\!(q^{1/n})\!)$) is by definition a section of $\mathcal{M}_{\mathrm{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!)}$ over $\mathbb{C}(\!(q^{1/n})\!)$. By the definition of the (2-)fiber product, such sections are given by isomorphism classes of pairs (c, φ), where c is a morphism

$$c : \operatorname{Spec}\mathbb{C}(\!(q^{1/n})\!) \to \mathcal{M}$$

and φ is an isomorphism

$$\varphi : c^*\mathcal{E}_{\mathcal{M}} \xrightarrow{\sim} \mathrm{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!) \qquad \text{over } \mathbb{C}(\!(q^{1/n})\!) \tag{9}$$

where here $c^*\mathcal{E}_{\mathcal{M}}$ is the elliptic curve corresponding to $p \circ c : \operatorname{Spec}\mathbb{C}(\!(q^{1/n})\!) \to \mathcal{M}(1)$. An isomorphism between $(c, \varphi), (c', \varphi')$ is an isomorphism $f : c \xrightarrow{\sim} c'$ in the fiber category $\mathcal{M}(\mathbb{C}(\!(q^{1/n})\!))$ such that $\varphi' \circ p(f) = \varphi$.

Thus, an oriented cusp is represented by a pair (c, φ), where we think of c as the underlying framed cusp, and φ is an "orientation", which picks out a choice of isomorphism with the Tate curve amongst the two possibilities $\{\varphi, [-1] \circ \varphi\}$. If $\mathcal{M}$ is a moduli stack of elliptic curves with level structures, then the set of oriented cusps (valued in $\mathbb{C}(\!(q^{1/n})\!)$) can be identified, via the isomorphism φ, with the set of $\mathcal{M}$-level structures on $\mathrm{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!)$.

Let M be the coarse scheme of $\mathcal{M}$. Since $\mathbb{C}(\!(q^{1/\infty})\!)$ is algebraically closed, the set of framed cusps are in bijection with the set of ways of filling in the dotted arrow in the diagram

$$\begin{array}{ccc} & & M \\ & \nearrow & \big\downarrow p \\ \operatorname{Spec} \mathbb{C}(\!(q^{1/\infty})\!) & \xrightarrow{\mathrm{Tate}(q)} & M(1) \end{array} \tag{10}$$

Note that the bottom map corresponds to the unique framed cusp "$i\infty$" of $\mathcal{M}(1)$. Let $\overline{M}$ be the smooth compactification of M, which comes with a canonical map to the coarse moduli scheme $\overline{M(1)}$ of $\overline{\mathcal{M}(1)}$. By properness, the choice of a cusp $|c|$ of $\mathcal{M}$ determines a unique point of $\overline{M} - M$. We call this point the corresponding cusp of $\overline{M}$, which we also denote by $|c|$. The *coarse width* of $|c|$ is by definition the ramification index at this point. If the width is equal to the coarse width, then the cusp is called *regular*. Otherwise, it is called *irregular*. One can verify that this agrees with the notion of (coarse) width and (ir)regular cusps defined in §5.3.2. Furthermore, the number of framed cusps lying over any cusp is equal to the coarse width of that cusp.

If we choose a uniformization $\mathcal{M}^{\mathrm{an}} = [\mathcal{H}/\Gamma]$, then Γ is determined up to conjugacy, so whether or not $-I \in \Gamma$ is independent of the choice of uniformization. It follows from the isomorphism $\mathcal{M}^{\mathrm{an}} = [\mathcal{H}/\Gamma]$ that:

Proposition 5.3.1. *Let $p : \mathcal{M} \to \mathcal{M}(1)$ be finite etale. Choose a uniformization $\mathcal{M}^{an} = [\mathcal{H}/\Gamma]$. Then the following are equivalent:*

(a) $-I \in \Gamma$

(b) There exists an object $x \in \mathcal{M}$ admitting an automorphism lying over the automorphism $[-1]$ of $p(x) \in \mathcal{M}(1)$

(c) Every object $x \in \mathcal{M}$ admits an automorphism lying over the automorphism $[-1]$ of $p(x) \in \mathcal{M}(1)$

For a framed cusp c (valued in $\mathbb{C}(\!(q^{1/n})\!)$) the set of pairs (c, φ) lying over c is a torsor under $\mathrm{Aut}_{\mathbb{C}(\!(q^{1/n})\!)}(\mathrm{Tate}(q)) = \{\pm 1\}$. In particular, this set is precisely $\{(c, \varphi), (c, [-1] \circ \varphi)\}$. These represent the same oriented cusp if and only if $[-1] \in \mathrm{Aut}_{\mathcal{M}}(c)$, which by the above occurs if and only if $-I \in \Gamma$. Thus, the number of oriented cusps lying over any framed cusp is 1 if $-I \in \Gamma$, and 2 otherwise. To summarize, we have:

Proposition 5.3.2. *Let $p : \mathcal{M} \to \mathcal{M}(1)$ be finite etale. Choose a uniformization $\mathcal{M} = [\mathcal{H}/\Gamma]$. Let c be a framed cusp of $\mathcal{M}$, and $|c|$ the underlying cusp. Let e denote the coarse width of $|c|$. The set of framed cusps lying over $|c|$ is a torsor under μ_e, and the number of oriented cusps lying over c is 1 if $-I \in \Gamma$, and 2 otherwise.*

5.3.4 Uniformization and cusps

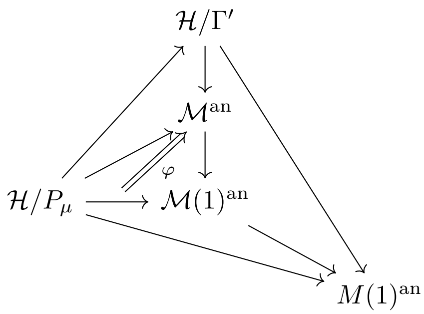
Here we will show that a choice of uniformization of $\mathcal{M}^{\mathrm{an}}$ determines an oriented cusp "$i\infty$" of $\mathcal{M}$.

Let us make a choice of uniformization of $\mathcal{M}^{\mathrm{an}} = [\mathcal{H}/\Gamma]$, and let μ be the cusp width of $i\infty$. Then $P_\mu = \langle \left[\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right] \rangle \subset \Gamma$, and we may further refine the uniformization diagram (8) as:

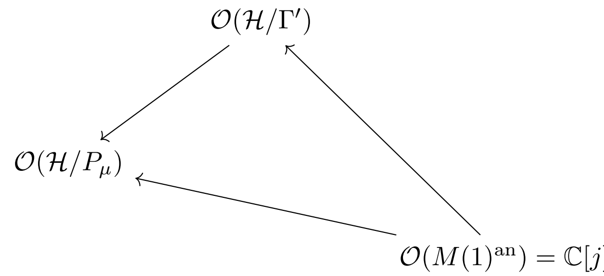
$$\begin{array}{ccccccc} & & \mathcal{H}/P_\mu & \longrightarrow & [\mathcal{H}/\Gamma] & \xrightarrow{\sim} & \mathcal{M}^{\mathrm{an}} \\ & \nearrow & \big\downarrow & & \big\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \big\downarrow p \\ \mathcal{H} & \longrightarrow & \mathcal{H}/P_1 & \longrightarrow & [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})] & \xrightarrow[\sim]{} & \mathcal{M}(1)^{\mathrm{an}} \end{array}$$

where the unmarked arrows are the canonical projections. The map $\mathcal{H}/P_1 \to [\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})] \to \mathcal{M}(1)^{\mathrm{an}}$ is given by the analytic Tate curve $\mathrm{Tate}^{\mathrm{an}}$, and hence the composition $\mathcal{H}/P_\mu \to \mathcal{M}(1)^{\mathrm{an}}$ is given by the analytic μ-sided Tate curve $\mathrm{Tate}^{\mathrm{an}}_\mu$ (c.f. §5.2.1). As in (8), the triangle on the left 2-commutes via the identity 2-isomorphism, and hence this 2-commutative diagram determines a "analytic oriented cusp" of $\mathcal{M}^{\mathrm{an}}$.

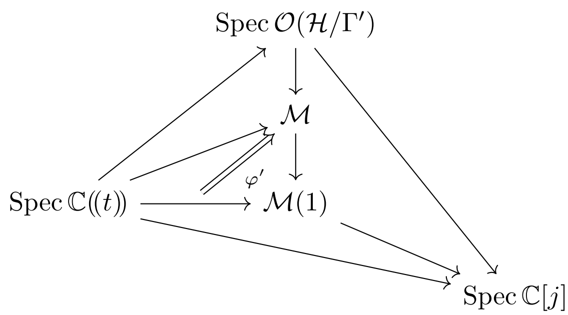
We now argue that this also determines an oriented cusp of $\mathcal{M}$, as follows. Let μ denote the cusp width of $i\infty$ on $[\mathcal{H}/\Gamma]$. Let $\Gamma' \subset \Gamma$ be a torsion-free finite index subgroup such that the cusp $i\infty$ of $\mathcal{H}/\Gamma'$ has the same width μ. For example, we may take $\Gamma' := \Gamma \cap \Gamma_1(5)$. Let $\mathcal{O}(\mathcal{H}/P_\mu)$ denote the ring of holomorphic functions on $\mathcal{H}/P_\mu$ which is meromorphic at "$i\infty$", and let $\mathcal{O}(\mathcal{H}/\Gamma')$ (resp. $\mathcal{O}(M(1)^{\mathrm{an}})$) be the ring of holomorphic functions on $\mathcal{H}/\Gamma'$ (resp. $M(1)^{\mathrm{an}} = \mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$) which are meromorphic at all cusps. Then, we have diagrams



whence a diagram



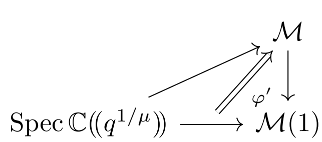
where the map $\mathcal{H}/\Gamma' \to \mathcal{M}^{\mathrm{an}}$ is the composition $\mathcal{H}/\Gamma' \to [\mathcal{H}/\Gamma] \to \mathcal{M}^{\mathrm{an}}$, the "central triangle" on the left 2-commutes via φ, and all other triangles 1-commute (2-commute via the identity 2-isomorphism). Let t be an isomorphism $t : \mathcal{H}/P_\mu \to D^\circ \subset \mathbb{C}$ sending "$i\infty$" to "0". Then, "expanding in t" defines a homomorphism $\mathcal{O}(\mathcal{H}/P_\mu) \to \mathbb{C}(\!(t)\!)$. There is a canonical choice of t, denoted by "$q^{1/\mu}$", which sends $P_\mu\tau \mapsto e^{2\pi i\tau/\mu} \in D^\circ$, and for any choice of t, we have $t = vq^{1/\mu}$ for some holomorphic function $v : \mathcal{H}/P_\mu \to \mathbb{C}^\times$ whose $q^{1/\mu}$-expansion lies in $\mathbb{C}[\![q^{1/\mu}]\!]^\times$. Since $\mathcal{H}/\Gamma'$ and $M(1)^{\mathrm{an}}$ are algebraic, we obtain a commutative diagram[14]



(11)

where the map $\operatorname{Spec}\mathbb{C}(\!(t)\!) \to \mathcal{M}(1)$ is the μ-sided Tate curve in the uniformizer t, and the 2-isomorphism φ' is the "algebraization" of φ, which exists because the commutativity of the outer triangle imply that the source and target of φ are both twists of the μ-sided Tate curve over $\mathcal{H}/P_\mu$, and two such twists are analytically isomorphic if and only if they are algebraically isomorphic viewed as elliptic curves over $\mathbb{C}(\!(t)\!)$. Thus, from the uniformization $u : \mathcal{H} \to \mathcal{M}^{\mathrm{an}}$, we have defined an oriented cusp of $\mathcal{M}$.

Definition 5.3.3. Given a choice of uniformization of $\mathcal{M}^{\mathrm{an}} = [\mathcal{H}/\Gamma]$ (given by the data $u : \mathcal{H} \to \mathcal{M}^{\mathrm{an}}$ and the 2-isomorphism φ), let μ be the cusp width of $i\infty$. Then, the associated oriented cusp "$i\infty$" is by definition the 2-commutative triangle



coming from in (11), where we have chosen $t := q^{1/\mu}$ to be the function $\mathcal{H}/P_\mu \xrightarrow{\sim} D^\circ$ given by $P_\mu\tau \mapsto e^{2\pi i\tau/\mu}$.

[14] As usual, all triangles without indicated 2-isomorphisms actually 1-commute.

6 Equivalence between Katz and classical modular forms

6.1 Katz modular forms

Let S be a noetherian scheme, and let $\mathcal{M}(1)$ be the moduli stack of elliptic curves over S. Let $\overline{\mathcal{M}(1)}$ be the moduli stack of stable (1,1)-curves (ie, stable 1-pointed curves of genus 1, c.f. [Knu83] §1). Then $\overline{\mathcal{M}(1)}$ is a smooth proper DM stack containing $\mathcal{M}(1)$ as an open substack. Let $\omega_{\overline{\mathcal{M}(1)}}$ be the functor which to any stable (1,1)-curve $q : E \to T$ associates the group $\Gamma(T, q_*\omega_q)$, where ω_q is the dualizing sheaf of q.

If q is smooth, then $\omega_q = \Omega^1_{E/T}$. If $T = \operatorname{Spec}\mathbb{C}$ and $E \to T$ is a nodal cubic, then ω_q can be identified with the subsheaf of the sheaf of meromorphic differentials on the normalization of E consisting of those which are holomorphic away from the preimages of the node, having at worst logarithmic poles at those preimages, and such that the residues at the preimages sum to 0 (c.f. [Man99], §V.1.1).

The sheaves $q_*\omega_q$ are invertible $\mathcal{O}_T$-modules and commute with arbitrary base change ([DR75] §II, Proposition 1.6). Thus, $\omega_{\overline{\mathcal{M}(1)}}$ defines an invertible sheaf on $\overline{\mathcal{M}(1)}$, called the *Hodge bundle*. In particular, it is a coherent sheaf.

Let $\omega_{\overline{\mathcal{M}(1)}^{\mathrm{an}}}$ be defined in exactly the same way. That is, for any $T \in \underline{\mathbf{An}}$ and analytic family of stable (1,1)-curves $q : E \to T$, $\omega_{\overline{\mathcal{M}(1)}^{\mathrm{an}}}$ associates to $E \to T$ the group $\Gamma(T, q_*\omega_q)$. Let $\omega_{\mathcal{M}(1)^{\mathrm{an}}}$ be the restriction of $\omega_{\overline{\mathcal{M}(1)}^{\mathrm{an}}}$ to the open substack $\mathcal{M}(1)^{\mathrm{an}} \subset \overline{\mathcal{M}(1)}^{\mathrm{an}}$.

Definition 6.1.1. Let $p : \mathcal{M} \to \mathcal{M}(1)$ be a finite etale morphism (still working over S). Suppose $\mathcal{M}$ is an open substack of a smooth proper DM stack $\overline{\mathcal{M}}$, and that p extends to a map between compactifications which we also call $p : \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$. A (weakly holomorphic) Katz modular form for $\mathcal{M}$ of weight k is a global section of $\omega_{\mathcal{M}}^{\otimes k} := p^*\omega_{\mathcal{M}(1)}^{\otimes k}$. A holomorphic Katz modular form for $\overline{\mathcal{M}}$ of weight k is a global section of $\omega_{\overline{\mathcal{M}}}^{\otimes k} := p^*\omega_{\overline{\mathcal{M}(1)}}^{\otimes k}$. By the description of pullbacks given in §2.1, for any morphism $T \to \overline{\mathcal{M}}$, if the composition $T \to \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$ corresponds to the stable curve $q : E \to T$, then we have

$$\omega_{\overline{\mathcal{M}}}^{\otimes k}(T \to \overline{\mathcal{M}}) = \omega_{\overline{\mathcal{M}(1)}}^{\otimes k}(T \to \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}) = \Gamma(T, q_*\omega_q)$$

Thus, the space of weakly holomorphic (resp. holomorphic) Katz modular forms for $\mathcal{M}$ (resp. $\overline{\mathcal{M}}$) of weight k is

$$H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) \qquad (\text{resp. } H^0(\overline{\mathcal{M}}, \omega_{\overline{\mathcal{M}}}^{\otimes k}))$$

If $S = \operatorname{Spec}\mathbb{C}$, the formation of the Hodge bundle commutes with analytification[15]. That is,

$$(\omega_{\overline{\mathcal{M}}})^{\mathrm{an}} = \omega_{\overline{\mathcal{M}}^{\mathrm{an}}}$$

Similarly, let $\omega_{\mathcal{M}^{\mathrm{an}}} := (p^{\mathrm{an}})^*\omega_{\mathcal{M}(1)^{\mathrm{an}}}$ and $\omega_{\overline{\mathcal{M}}^{\mathrm{an}}} := (p^{\mathrm{an}})^*\omega_{\overline{\mathcal{M}(1)}^{\mathrm{an}}}$.

6.2 Classical modular forms

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup. Let $\mathcal{H}$ denote the upper half plane.

Definition 6.2.1. A classical (weakly holomorphic) modular form for Γ of weight k is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

1. f is modular of weight k for Γ. That is, for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, we have $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\tau \in \mathcal{H}$.
2. f is meromorphic at the cusps.

If f is moreover holomorphic at the cusps, then we call f a holomorphic modular form. A modular form of weight $k = 0$ is called a modular function. The space of weakly holomorphic (resp. holomorphic) modular forms for Γ of weight k is denoted

$$M_k(\Gamma) \qquad (\text{resp. } M_k^{\mathrm{hol}}(\Gamma))$$

[15] This is because elliptic curves are proper, and hence all analytic sections of the dualizing sheaf are algebraic.

Note that since Γ is finite index inside $\mathrm{SL}_2(\mathbb{Z})$, for some integer μ, we must have $\left[\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$. The smallest positive integer μ satisfying this property is called the cusp width of $i\infty$ relative to Γ. By property (1), we have:

$$f(\tau + \mu) = f(\tau) \qquad \text{for all } \tau \in \mathcal{H}$$

In particular, f is μ-periodic, and hence has a Fourier expansion as a Laurent series in $q^{1/\mu} := e^{2\pi i/\mu}$.

6.3 From Katz to classical

In this section we return to setting $S = \operatorname{Spec}\mathbb{C}$.

Associated to the family $\mathbb{E}/\mathcal{H}$, we also have the Hodge bundle $\omega_{\mathcal{H}}$ on $\mathcal{H}$, which for every $T \in \underline{\mathbf{An}}$ and every map $T \to \mathcal{H}$ we associate the global holomorphic differentials on $\mathbb{E}_T$. Since $\mathcal{H}$ is contractible, this bundle is trivial, and is specifically trivialized by the section "dz" which to every $\tau \in \mathcal{H}$ associates the differential dz on $\mathbb{E}_\tau = \mathbb{C}/\Lambda_\tau$. Similarly, $dz^{\otimes k}$ is a nowhere vanishing section of $\omega_{\mathcal{H}}^{\otimes k}$.

Let $\mathrm{pt} \in \underline{\mathbf{An}}$ denote the 1-point space (with structure sheaf the constant sheaf $\mathbb{C}$)

Let $p : \mathcal{M} \to \mathcal{M}(1)$ be a finite etale morphism of connected DM stacks, extending to a map $p : \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$ of smooth proper stacks. Let G be a (weakly holomorphic) Katz modular form for $\mathcal{M}$ of weight k - that is, a global section of $\omega_{\mathcal{M}}^{\otimes k}$.

As in §5.3, let us choose a uniformization (e.g., a diagram as in (7)), consisting of a morphism $u : \mathcal{H} \to \mathcal{M}^{\mathrm{an}}$ and a 2-isomorphism φ witnessing the 2-commutativity of the diagram. As in (8), u factors uniquely through an equivalence $[\mathcal{H}/\Gamma] \cong \mathcal{M}^{\mathrm{an}}$, which determines a finite index subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.

For any $\tau \in \mathcal{H}$, $u(\tau)$ gives us an object of $\mathcal{M}^{\mathrm{an}}(\mathrm{pt})$ whose image $p(u(\tau))$ in $\mathcal{M}(1)^{\mathrm{an}}(\mathrm{pt})$ is an elliptic curve, which via φ is equipped with an isomorphism

$$\varphi_\tau : \mathbb{E}_\tau \xrightarrow{\sim} p(u(\tau))$$

We may evaluate G^{an} at $u(\tau)$ to obtain a k-fold differential on $p(u(\tau))$, and the pullback $\varphi_\tau^* G^{\mathrm{an}}(u(\tau))$ must be $\lambda_\tau (2\pi i dz)^{\otimes k}$ for some $\lambda_\tau \in \mathbb{C}$. We define the classical modular form f_G associated to G as:

$$f_G : \mathcal{H} \to \mathbb{C} \qquad f_G(\tau) := \lambda_\tau \tag{12}$$

In other words, we have $f_G(\tau) = \varphi_\tau^* G^{\mathrm{an}}(u(\tau))/(2\pi i dz)^{\otimes k}$.

Proposition 6.3.1. *The function f_G defined above is holomorphic (on $\mathcal{H}$) and is weight k modular for Γ.*

Proof. First we show that it is holomorphic on $\mathcal{H}$. The algebraic section G defines an analytic section G^{an} of $(\omega_{\mathcal{M}}^{\otimes k})^{\mathrm{an}} = \omega_{\mathcal{M}^{\mathrm{an}}}^{\otimes k}$, which gives a nowhere vanishing differential $G^{\mathrm{an}}(u(\mathcal{H}))$ of the elliptic curve $p(u(\mathcal{H}))$ over $\mathcal{H}$, and our choice of uniformization gives an isomorphism of elliptic curves over $\mathcal{H}$

$$\varphi_{\mathcal{H}} : \mathbb{E} \xrightarrow{\sim} p(u(\mathcal{H}))$$

The function f_G is just the quotient of $\varphi_{\mathcal{H}}^* G^{\mathrm{an}}(u(\mathcal{H}))$ and the nowhere vanishing holomorphic section $(2\pi i dz)^{\otimes k}$ of the holomorphic vector bundle $\omega_{\mathcal{H}}$, and hence is holomorphic (on $\mathcal{H}$).

Next we show that f_G behaves as expected under Γ. Suppose $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, then γ determines an isomorphism $\tilde{\gamma}_\tau$

$$\tilde{\gamma}_\tau : \mathbb{E}_\tau \to \mathbb{E}_{\gamma\tau} \qquad \gamma(z + \Lambda_\tau) = \frac{1}{c\tau + d} z + \Lambda_{\gamma\tau}$$

Let $u(\gamma_\tau)$ denote the image of the isomorphism $\gamma_\tau : \tau \to \gamma\tau$ in $[\mathcal{H}/\Gamma]$ under the equivalence $[\mathcal{H}/\Gamma] \xrightarrow{\sim} \mathcal{M}^{\mathrm{an}}$. The fact that G^{an} is a global section of $\omega_{\mathcal{M}^{\mathrm{an}}}$ then implies (c.f. Example 3.3.1) that

$$G^{\mathrm{an}}(u(\tau)) = u(\gamma_\tau)^* G^{\mathrm{an}}(u(\gamma\tau)) = p(u(\gamma_\tau))^* G^{\mathrm{an}}(u(\gamma\tau))$$

where on the right side of the second equality we are viewing $G^{\mathrm{an}}(u(\gamma\tau))$ as a differential on the elliptic curve $p(u(\gamma\tau))$. The fact that φ witnesses the 2-commutativity of the diagram (7) implies that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{E}_\tau & \xrightarrow{\varphi_\tau} & p(u(\tau)) \\ \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle p(u(\gamma_\tau))} \\ \mathbb{E}_{\gamma\tau} & \xrightarrow{\varphi_{\gamma\tau}} & p(u(\gamma\tau)) \end{array}$$

From this, we get:

$$f_G(\tau)(2\pi i dz|_{\mathbb{E}_\tau})^{\otimes k} = \varphi_\tau^* G^{\mathrm{an}}(u(\tau)) = \varphi_\tau^* p(u(\gamma_\tau))^* G^{\mathrm{an}}(u(\gamma\tau)) = \tilde{\gamma}^* \varphi_{\gamma\tau}^* G^{\mathrm{an}}(u(\gamma\tau))$$

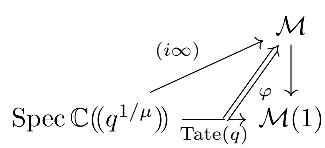
Applying $(\tilde{\gamma}^{-1})^*$ to both sides yields

$$(\tilde{\gamma}^{-1})^* f_G(\tau)(2\pi i dz|_{\mathbb{E}_\tau})^{\otimes k} = (c\tau + d)^k f_G(\tau)(2\pi i dz|_{\mathbb{E}_{\gamma\tau}})^{\otimes k} = \varphi_{\gamma\tau}^* G^{\mathrm{an}}(u(\gamma\tau))$$

but the last equality gives us precisely that $f_G(\gamma\tau) = (c\tau + d)^k f_G(\tau)$, as desired. □

We wish to show that f_G is meromorphic at the cusps. For this, it suffices to show that its q-expansion at $i\infty$ lies in $\mathbb{C}(\!(q^{1/\mu})\!)$. This is a consequence of the fact that we can recover q-expansions algebraically by evaluating at oriented cusps (c.f. 5.3):

Proposition 6.3.2. *Let $\mathcal{M} \to \mathcal{M}(1)$ be finite etale and G a global section of $\omega_{\mathcal{M}}^{\otimes k}$. Let us choose a uniformization of $\mathcal{M}^{an} = [\mathcal{H}/\Gamma]$. By 5.3.3, this choice of uniformization defines an oriented cusp "$((i\infty), \varphi)$" of $\mathcal{M}$ - ie, a diagram*



Let μ denote the cusp width of $(i\infty)$. Let q-exp(f_G) denote the $q^{1/\mu} = e^{2\pi i\tau/\mu}$-expansion of the holomorphic function $f_G : \mathcal{H} \to \mathbb{C}$. Then, q-exp$(f_G)$ is given by the formula

$$\varphi^* G((i\infty)) = q\text{-}exp(f_G)\omega_{can}^{\otimes k} \tag{13}$$

in the 1-dimensional $\mathbb{C}(\!(q^{1/\mu})\!)$-vector space $\Gamma(\mathrm{Tate}(q), \Omega^1_{\mathrm{Tate}(q)/\mathbb{C}(\!(q^{1/\mu})\!)})$. In particular, q-exp$(f_G)$ is a finite-tailed Laurent series, hence is meromorphic at $i\infty$. Let $d := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Let $\{\gamma_i\}_{i=1,\ldots,d}$ be representatives of the cosets $\mathrm{SL}_2(\mathbb{Z})/\Gamma$, and let μ_i be the cusp width of $\gamma_i(i\infty)$. Then $\mathcal{M}$ has d-oriented cusps, and the formulas (13) associated to each oriented cusp gives the expansions of f in $e^{2\pi i \gamma_i^{-1}\tau/\mu_i}$ (a uniformizer at $\gamma(i\infty)$) for $i = 1, \ldots, d$.

Proof. The statement up through (13) follows from the definition of f_G and the construction of the oriented cusp "$i\infty$" in 5.3.3 associated to our choice of uniformization. The second statement about q-expansions at other cusps is straightforward to check. □

Remark 6.3.3. Since the passage from Katz to classical involves evaluating the global section G of $\omega_{\mathcal{M}}^{\otimes k}$ on individual elliptic curves over $\mathbb{C}$, one might wonder why the Tate curve over $\mathbb{C}(\!(q^{1/\mu})\!)$ plays such a distinguished role as compared to its twist, since they define analytic families which are fiberwise isomorphic. In this setting the distinguishing characteristic of the Tate curve is its access to the differential ω_{can}, which coincides with the differential $2\pi i dz \in \Omega^1_{\mathbb{E}/\mathcal{H}}$ on all of $\mathcal{H}$. Any choice of a holomorphic differential on the twist of the Tate curve over $\mathbb{C}(\!(q^{1/\mu})\!)$ will only agree with $2\pi i dz$ on "alternating" vertical strips in $\mathcal{H}$ of width μ. On the other strips it will correspond to $-2\pi i dz$.

6.4 From classical to Katz

Let $f : \mathcal{H} \to \mathbb{C}$ be a *holomorphic* modular form for a finite index subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. That is to say, f satisfies the conditions of 6.2.1 and is moreover holomorphic at all cusps. Assume that there is a morphism of connected smooth proper (algebraic) DM stacks $p : \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$ such that the restriction of p to the open substack $\mathcal{M}(1)$ gives a (representable) finite etale morphism $\mathcal{M} \to \mathcal{M}(1)$ whose analytification can be identified with $[\mathcal{H}/\Gamma] \to [\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})]$ as in §5.3.

It follows from the results of [BR11] that such compactifications $\overline{\mathcal{M}}$ always exist, though in general the map $p : \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$ will not be representable.

We wish to construct, using f, a global section G_f of $\omega_{\overline{\mathcal{M}}}^{\otimes k} := p^*\omega_{\overline{\mathcal{M}(1)}}^{\otimes k}$. The first step is to construct an analytic section G_f^{an} of $\omega_{\overline{\mathcal{M}}^{\text{an}}}^{\otimes k}$.

Let $q : \mathbb{E} \to \mathcal{H}$ be the "universal framed elliptic curve over $\mathcal{H}$" as before. Note that $f(\tau)dz^{\otimes k}$ is a holomorphic section of $\omega_{\mathcal{H}}^{\otimes k} = q_*(\Omega^1_{\mathbb{E}/\mathcal{H}})^{\otimes k}$. More precisely, to every point $\tau \in \mathcal{H}$, f associates the k-fold holomorphic differential $f(\tau)(2\pi i dz)^{\otimes k}$ on $\mathbb{E}_\tau$. In order to show that the same rule defines a section of $\omega_{[\mathcal{H}/\Gamma]}^{\otimes k}$, we must check that for every $\tau \in \mathcal{H}$ and $\gamma \in \Gamma$ inducing the isomorphism

$$\gamma : \mathbb{E}_\tau \to \mathbb{E}_{\gamma\tau} \qquad \gamma(z + \Lambda_\tau) = \frac{1}{c\tau + d} z + \Lambda_{\gamma\tau}$$

we have $\gamma^* f(\gamma\tau)(2\pi i dz)^{\otimes k} = f(\tau)(2\pi i dz)^{\otimes k}$. Indeed, by the modular property of f, we have:

$$\gamma^* f(\gamma\tau)(2\pi i dz)^{\otimes k} = f(\gamma\tau)\gamma^*(2\pi i dz)^{\otimes k} = f(\gamma\tau)(c\tau+d)^{-k}(2\pi i dz)^{\otimes k} = f(\tau)(2\pi i dz)^{\otimes k}$$

Thus, we will define:

$$G_f^{\text{an}}(\mathbb{E}_\tau) := f(\tau)(2\pi i dz)^{\otimes k} \qquad \tau \in \mathcal{H}$$

Now we wish to show that the section $G_f^{\text{an}}(\mathbb{E}_\tau) := f(\tau)(2\pi i dz)^{\otimes k}$ for $\tau \in \mathcal{H}$ extends to the cusps. It will suffice to make the argument for the cusp $i\infty$, as the procedure for other cusps is the same.

Since $\overline{\mathcal{M}}$ is assumed Deligne-Mumford, the cusp $i\infty$ of $\overline{\mathcal{M}}^{\text{an}}$ admits an etale neighborhood $V \to \overline{\mathcal{M}}^{\text{an}}$ with $V \in \underline{\mathbf{An}}$. Since $\overline{\mathcal{M}}$ is separated, by shrinking V, we may assume that there is a unique point $v_0 \in V$ whose image in $\overline{\mathcal{M}(1)}^{\text{an}}$ corresponds to a singular curve. Let $V^\circ := V - \{v_0\}$. By the discussion in §5.2, the Tate curve defines an analytic family Tate^{an} of stable (1,1) curves over the open unit disk $D \subset \mathbb{C}$ whose only singular fiber lies at $0 \in D$. Thus, $\text{Tate}^{\text{an}}/D$ defines an *etale* morphism $D \to \overline{\mathcal{M}(1)}$ sending $0 \mapsto i\infty$ (c.f. 5.2.5). By possibly replacing V with $V \times_{\overline{\mathcal{M}(1)}^{\text{an}}} D$, we may moreover assume that we have a commutative diagram[16]

$$\begin{array}{ccc} V & \xrightarrow{\text{ét}} & \overline{\mathcal{M}}^{\text{an}} \\ \downarrow{\scriptstyle p|_V} & & \downarrow{\scriptstyle p} \\ D & \xrightarrow{\text{Tate}^{\text{an}}} & \overline{\mathcal{M}(1)}^{\text{an}} \end{array} \tag{14}$$

Because $\overline{\mathcal{M}}^{\text{an}}$ is smooth, V is smooth, and hence by further shrinking V around v_0, we may assume that V is biholomorphic to a connected open subset of $\mathbb{C}$. Since $v_0 \in V$ is the only point corresponding to a singular curve, the map $V \to D$ is nonconstant, hence open, and hence by replacing V with a subset $V' \subset V$, and replacing D with $D' \subset D$, we may assume that $p|_{V'} : V' \to D'$ is a surjection between open disks of positive radius with finite fibers. Since $\mathcal{M}^{\text{an}} \to \mathcal{M}(1)^{\text{an}}$ is etale, the restriction of $p|_{V'}$ to V'° gives an etale morphism $V'^\circ \to D'^\circ$, which being a connected unramified cover of an open punctured disk, must be given by $z \mapsto z^n$ for some $n \geq 1$. Thus, if q is a parameter on D'°, then $q^{1/n}$ is a parameter on V'° and the map $V'^\circ \to U'^\circ \to \overline{\mathcal{M}(1)}^{\text{an}}$ corresponds to the pullback $\text{Tate}^{\text{an}}|_{V'^\circ}$ of Tate^{an} to V'°. Thus $\text{Tate}^{\text{an}}|_{V'^\circ}$ is a restriction of the n-sided Tate curve $\text{Tate}_n^{\text{an}}$ to a small punctured disk. By 5.2.2, this implies that $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$. In particular, f must be n-periodic. Similarly, the map $V' \to U' \to \overline{\mathcal{M}(1)}^{\text{an}}$ is given by $\text{Tate}_n^{\text{an}}|_{V'}$, which is a stable family of $(1,1)$-curves over V', and the same computation as in 5.2.4 shows that ω_{can} extends to give a global section of the dualizing sheaf for this family.

Since f is n-periodic and holomorphic at $i\infty$, it lifts to a holomorphic function on the disk V' with parameter $q^{1/n}$. Thus, at $v \in V'$ we can define a section G_f^{an} of $\omega_{\overline{\mathcal{M}}^{\text{an}}}^{\otimes k}$ by

$$G_f^{\text{an}}(\text{Tate}_n^{\text{an}}|_{q^{1/n}=v}) := (2\pi i)^{-k} f(v)\omega_{\text{can}}^{\otimes k}$$

[16] We cannot assume that this is a pullback diagram, since in general p may not be a representable morphism.

By the formulas in §5.2, this agrees with our definition $G_f^{\text{an}}(\mathbb{E}_\tau)$ for $\tau \in \mathcal{H}$ and hence doing this for every cusp, we have constructed a global analytic section G_f^{an} of $\omega_{\overline{\mathcal{M}}^{\text{an}}}^{\otimes k}$, which defines a morphism

$$G_f^{\text{an}} : \mathcal{O}_{\overline{\mathcal{M}}^{\text{an}}} \longrightarrow \omega_{\overline{\mathcal{M}}^{\text{an}}}^{\otimes k}$$

Since $\overline{\mathcal{M}}$ is proper, by GAGA (c.f. §4.2), analytification induces an equivalence of categories an : $\underline{\textbf{Coh}}(\overline{\mathcal{M}}) \cong \underline{\textbf{Coh}}(\overline{\mathcal{M}}^{\text{an}})$, and hence the morphism G_f^{an} above corresponds to a morphism

$$G_f : \mathcal{O}_{\overline{\mathcal{M}}} \to \omega_{\overline{\mathcal{M}}}^{\otimes k}$$

which is the desired global section of $\omega_{\overline{\mathcal{M}}}^{\otimes k}$.

6.5 The equivalence

Definition 6.5.1. For a subgroup $\Gamma \leq \text{SL}_2(\mathbb{Z})$, let $M_k(\Gamma)$ denote the $\mathbb{C}$-vector space of weakly holomorphic modular forms of weight k for Γ, and $M_k^{\text{hol}}(\Gamma)$ its subspace of holomorphic forms.

We have essentially proven the following result:

Theorem 6.5.2. *Let $p : \mathcal{M} \to \mathcal{M}(1)$ be a finite etale morphism (all taken over $\mathbb{C}$), with $\mathcal{M}$ connected. Let $\overline{\mathcal{M}}$ be a smooth compactification of $\mathcal{M}$ which is Deligne-Mumford, and such that p extends to a morphism $p : \overline{\mathcal{M}} \to \overline{\mathcal{M}(1)}$. Choose a uniformization $\mathcal{M}^{an} \cong [\mathcal{H}/\Gamma]$ as in §5.3. Then, we have an isomorphism of vector spaces (depending on our choice of uniformization)*

$$H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) \cong M_k(\Gamma)$$

restricting to an isomorphism

$$H^0(\overline{\mathcal{M}}, \omega_{\overline{\mathcal{M}}}^{\otimes k}) \cong M_k^{hol}(\Gamma)$$

given by the mutually inverse maps $G \mapsto f_G$ for $G \in H^0(\overline{\mathcal{M}}, \omega_{\overline{\mathcal{M}}}^{\otimes k})$ and $f \mapsto G_f$ for $f \in M_k(\Gamma)$ as described in §6.3 and §6.4.

Proof. The fact that $f \mapsto G_f$ and $G \mapsto f_G$ are mutually inverse is essentially clear from definition. The discussion above certainly proves the second isomorphism. To prove the first, we can use the same argument, but using dualizing sheaves with bounded poles at the cusps $\omega_{\overline{\mathcal{M}}}^{\otimes k}(D)$ for an appropriate choice of cuspidal divisor D. These are coherent, and so the same GAGA argument applies. We omit the details. □

7 Arithmetic considerations

7.1 Base change and the q-expansion principle

The main result 7.1.4 is analogous to the classical q-expansion principle (c.f. Katz [Kat73] §1.6). However, unlike in Katz [Kat73], which restricted to the case of proper modular schemes and used Grothendieck's comparison theorem for formal schemes (which only holds in the proper setting), here we obtain a slightly more general result by adapting an argument of Brian Conrad in his lecture notes *the algebraic theory of q-expansions* to apparently remove the properness assumption. This seems useful for giving an analytic description of arithmetic models for stacks finite etale over $\mathcal{M}(1)$ over some Dedekind ring $\mathcal{O}$ (c.f. 7.3.2).

In this section by default we work over $S = \operatorname{Spec} \mathcal{O}$ where $\mathcal{O}$ is a Noetherian ring.

Let $\mathcal{M}(1)$ (resp. $\overline{\mathcal{M}(1)}$) be the moduli stack of elliptic curves (stable (1,1)-curves) over $\mathcal{O}$. Suppose we have a finite etale morphism of DM stacks $p : \mathcal{M} \to \mathcal{M}(1)$ over $\mathcal{O}$, then we have a notion of a meromorphic Katz modular form over $\mathcal{O}$ - that is, an element of the $\mathcal{O}$-module $H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) := H^0(\mathcal{M}, p^*\omega^{\otimes k})$.

Theorem 7.1.1 (Flat base change)**.** *Let B be a flat $\mathcal{O}$-algebra. There is a canonical isomorphism*

$$H^0(\mathcal{M}_B, \omega_{\mathcal{M}_B}^{\otimes k}) \cong H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) \otimes_{\mathcal{O}} B$$

Proof. The point is that global sections can be computed as a kernel (c.f. §3.3), and flat base change preserves kernels. Specifically, let $U \to \mathcal{M}$ be a etale covering of $\mathcal{M}$, then $U_B \to \mathcal{M}_B$ is also an etale covering. Since B is $\mathcal{O}$-flat, we have a commutative diagram with exact rows (exactness follows from §3.3)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) & \longrightarrow & \omega_{\mathcal{M}}^{\otimes k}(U) & \longrightarrow & \omega_{\mathcal{M}}^{\otimes k}(U \times_{\mathcal{X}} U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) \otimes_{\mathcal{O}} B & \longrightarrow & \omega_{\mathcal{M}}^{\otimes k}(U) \otimes_{\mathcal{O}} B & \longrightarrow & \omega_{\mathcal{M}}^{\otimes k}(U \times_{\mathcal{X}} U) \otimes_{\mathcal{O}} B \end{array}$$

Since U, U_B, $U \times_{\mathcal{X}} U$ and $(U \times_{\mathcal{X}} U)_B = U_B \times_{\mathcal{X}_B} U_B$ are schemes, and the restriction of $\omega_{\mathcal{M}}^{\otimes k}$ to a scheme is an ordinary quasicoherent sheaf (§3.2), by the usual flat base change for quasicoherent sheaves on schemes, we find that the last two entries of the bottom row are just $\omega_{\mathcal{M}_B}^{\otimes k}(U_B) \to \omega_{\mathcal{M}_B}^{\otimes k}(U_B \times_{\mathcal{X}_B} U_B)$. The exactness then implies our desired isomorphism.

□

Corollary 7.1.2. *Given a flat morphism $\eta : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathcal{O}$, if $\mathcal{M}$ is geometrically connected, then let us choose a uniformization $\mathcal{M}_{\mathbb{C}}^{an} \cong [\mathcal{H}/\Gamma]$ as in §5.3. Then, we have an isomorphism*

$$M_k(\Gamma) \cong H^0(\mathcal{M}_{\mathbb{C}}, \omega_{\mathcal{M}_{\mathbb{C}}}^{\otimes k}) \cong H^0(\mathcal{M}, \omega_{\mathcal{M}}^{\otimes k}) \otimes_{\mathcal{O}} \mathbb{C}$$

Proof. Follows directly from 6.5.2 and 7.1.1. □

Due to this equivalence, from now on we will write Katz modular forms also using the letter "f".

Given an elliptic curve E over an S-scheme T corresponding to a morphism $E/T : T \to \mathcal{M}(1)$, the pullback

$$\begin{array}{ccc} \mathcal{M}(E/T) := \mathcal{M} \times_{\mathcal{M}(1)} T & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow p \\ T & \xrightarrow{E/T} & \mathcal{M}(1) \end{array} \tag{15}$$

is finite etale over T, and is called the "scheme of abstract $\mathcal{M}$-level structures on E/T". The sections of $\mathcal{M}(E/T) \to T$ are called "(abstract) $\mathcal{M}$-level structures on E/T" and any such section determines a morphism $T \to \mathcal{M}$ lifting $E/T : T \to \mathcal{M}(1)$ via p. If E is the Tate curve defined over some $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra Ω, then we will sometimes call an $\mathcal{M}$-level structure on $\operatorname{Tate}(q)/\Omega$ an *oriented cusp of $\mathcal{M}$ with values in* Ω.

Now let K be any $\mathcal{O}$-module, and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{M}$. Let $K \otimes_{\mathcal{O}} \mathcal{F}$ denote the sheaf on $\mathcal{M}$ associated to the presheaf of $\mathcal{O}_{\mathcal{M}}$-modules defined by the rule:

$$(U \to \mathcal{M}) \mapsto K \otimes_{\mathcal{O}} \mathcal{F}(U \to \mathcal{M}) \tag{16}$$

If U is an affine scheme, then the restriction to $U \to \mathcal{M}$ of the presheaf given by (16) defines a quasicoherent sheaf on U. Since restriction commutes with sheafification ([Sta16] 00WY), this implies that $(K \otimes_{\mathcal{O}} \mathcal{F})(U \to \mathcal{M}) = K \otimes_{\mathcal{O}} \mathcal{F}(U \to \mathcal{M})$ for any morphism $U \to \mathcal{M}$ with U an affine scheme.

Definition 7.1.3 (q-expansion)**.** Let K be any $\mathcal{O}$-module. Let Ω be a $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra, and let α be an $\mathcal{M}$-level structure on $\operatorname{Tate}(q)/\Omega$. From the above discussion, setting $\mathcal{F} = \omega_{\mathcal{M}}^{\otimes k}$ and $U = \operatorname{Spec} \Omega$ with map $\operatorname{Spec} \Omega \to \mathcal{M}$ given by $(\operatorname{Tate}(q)/\Omega, \alpha)$, we get a map

$$H^0(\mathcal{M}, K \otimes_{\mathcal{O}} \omega_{\mathcal{M}}^{\otimes k}) \to H^0(\operatorname{Spec} \Omega, K \otimes_{\mathcal{O}} \omega_{\operatorname{Tate}(q)/\Omega}^{\otimes k}) = K \otimes_{\mathcal{O}} H^0(\operatorname{Spec} \Omega, \omega_{\operatorname{Tate}(q)/\Omega}^{\otimes k})$$

and hence by taking quotients with the canonical differential $\omega_{\mathrm{can}}^{\otimes k}$ on $\operatorname{Tate}(q)/\Omega$, we obtain a map

$$H^0(\mathcal{M}, K \otimes_{\mathcal{O}} \omega_{\mathcal{M}}^{\otimes k}) \to K \otimes_{\mathcal{O}} \Omega \tag{17}$$

which is called "taking q-expansions at the oriented cusp $(\operatorname{Tate}(q)/\Omega, \alpha)$".

Theorem 7.1.4. *Let $\mathcal{O}$ be a Dedekind domain with $p \in \mathcal{O}^\times$ for some prime p. Let $\mathcal{M}$ a connected*[17] *stack finite etale over $\mathcal{M}(1) := \mathcal{M}(1)_{\mathcal{O}}$, and Ω an $\mathcal{O}$-flat $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra satisfying:*

($\star$) *The map $\operatorname{Spec}\Omega \longrightarrow M(1)_{\mathcal{O}} = \operatorname{Spec}\mathcal{O}[j]$ given by the Tate curve is dominant when restricted to every fiber over $\mathcal{O}$.*

Suppose there exists an $\mathcal{M}$-level structure α on $\operatorname{Tate}(q)/\Omega$, then the associated q-expansion map (17) is injective for any $\mathcal{O}$-module K.[18] *For examples of Ω satisfying ($\star$), see 7.1.7.*

Proof. Let $n := p^2$. By taking a connected component of the fiber product over $\mathcal{M}(1)$ with the $\Gamma_1(n)$-moduli stack, $\mathcal{M}$ admits a finite etale cover by an irreducible representable stack $\mathcal{M}'$. Since there exists a $\Gamma_1(n)$-structure over $\mathbb{Z}(\!(q^{1/n})\!)$, we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{M}', K \otimes \omega_{\mathcal{M}'}^{\otimes k}) & \longrightarrow & K \otimes_{\mathcal{O}} \Omega[q^{1/n}] \\ \uparrow & & \uparrow \\ H^0(\mathcal{M}, K \otimes \omega^{\otimes k}) & \longrightarrow & K \otimes_{\mathcal{O}} \Omega \end{array}$$

where the top horizontal map is given by the product of the $\Gamma_1(n)$-structure and α. Furthermore, the left vertical map is injective since $\mathcal{M}' \to \mathcal{M}$ is a covering in the etale topology. Note that $\Omega[q^{1/n}] := \Omega[x]/(x^n - q)$ is faithfully flat over Ω, and hence it also satisfies ($\star$). Thus, to prove the injectivity of the bottom horizontal map, it suffices to prove that of the top horizontal map, and hence we may assume $\mathcal{M}$ an integral affine scheme, smooth over $\mathcal{O}$.

Writing K as the filtered colimit of its finitely generated submodules, since colimits of injective maps are injective and cohomology on quasicompact quasiseparated schemes commutes with filtered colimits ([Sta16] 01FF), we may assume that K is $\mathcal{O}$-finite.

If $0 \to K' \to K \to K'' \to 0$ is an exact sequence of $\mathcal{O}$-modules, then tensoring with the flat $\mathcal{O}$-module $\omega^{\otimes k}$ and taking global sections on the top row, and tensoring with the $\mathcal{O}$-flat Ω on the bottom row yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{M}, K' \otimes_{\mathcal{O}} \omega^{\otimes k}) & \longrightarrow & H^0(\mathcal{M}, K \otimes_{\mathcal{O}} \omega^{\otimes k}) & \longrightarrow & H^0(\mathcal{M}, K'' \otimes_{\mathcal{O}} \omega^{\otimes k}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' \otimes_{\mathcal{O}} \Omega & \longrightarrow & K \otimes_{\mathcal{O}} \Omega & \longrightarrow & K'' \otimes_{\mathcal{O}} \Omega & \longrightarrow & 0 \end{array}$$

where the vertical maps are given by q-expansion. A diagram chase shows that the result for K' and K'' implies the result for K, and hence by considering the sequence $0 \to K_{\text{tors}} \to K \to K/K_{\text{tors}} \to 0$, we may assume that K is either torsion or torsion-free. In the torsion-free case, since $\omega^{\otimes k}$ is $\mathcal{O}$-flat, by [Sta16] 0AUU, we may choose an injection $K \hookrightarrow \mathcal{O}^{\oplus r}$, which induces an injection on global sections $H^0(\mathcal{M}, K \otimes_{\mathcal{O}} \omega^{\otimes k}) \hookrightarrow H^0(\mathcal{M}, \mathcal{O}^{\oplus r} \otimes_{\mathcal{O}} \omega^{\otimes k})$, and hence the torsion-free case is first reduced to the case $K = \mathcal{O}$, and using the flat injection $\mathcal{O} \hookrightarrow \operatorname{Frac}\mathcal{O}$, then reduced to the case $K = \operatorname{Frac}\mathcal{O}$. In the torsion case, since K is finite over the Noetherian $\mathcal{O}$, K_{tors} is also $\mathcal{O}$-finite, so $\operatorname{Ann}_{\mathcal{O}}(K)$ is a nonzero ideal, and hence K is finite over the Artinian ring $\mathcal{O}/\operatorname{Ann}_{\mathcal{O}}(K)$, so K is itself Artinian, and hence has a (finite) composition series[19]. Thus, examining the simple composition factors, we are reduced to the case $K \cong \mathcal{O}/\mathfrak{m}$ (as $\mathcal{O}$-modules) for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}$.

Since the ring structure on K is irrelevant, we are reduced to treating the case where K is a field, either $\operatorname{Frac}\mathcal{O}$ or $\mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. In this case, let $\mathcal{M}_K$ denote the base change of $\mathcal{M}$ by the map $\operatorname{Spec} K \to \operatorname{Spec}\mathcal{O}$. Since the base change map is affine, we have

$$H^0(\mathcal{M}, K \otimes_{\mathcal{O}} \omega^{\otimes k}) = H^0(\mathcal{M}_K, \omega_{\mathcal{M}_K}^{\otimes k})$$

[17]note that we do *not* require that $\mathcal{M}$ be geometrically connected, though I'm not sure what this buys us.

[18]If a connected component of a geometric fiber of $\mathcal{M}$ is a nontrivial finite etale cover of the corresponding geometric fiber of $\mathcal{M}(1)$, then the condition that there exists a prime $p \in \mathcal{O}^\times$ should be superfluous. Indeed, I believe for any Dedekind domain $\mathcal{O}$ having all primes as residue characteristics, it should be the case that $\pi_1(\mathcal{M}(1)_{\mathcal{O}}) = \pi_1(\operatorname{Spec}\mathcal{O})$, though I don't currently have a proof.

[19]This seems to be the only case where the Dedekind hypothesis on $\mathcal{O}$ is relevant (without it, $\mathcal{O}/\operatorname{Ann}_{\mathcal{O}}(K)$ may not be Artinian, and hence might not have a finite composition series. I wonder if the theorem is true when $\mathcal{O}$ is only assumed a Noetherian domain.

Since the map $\mathcal{M} \to M(1)_{\mathcal{O}}$ is faithfully flat, by property $(\star)$, the image of the Tate curve map $\operatorname{Spec} K \otimes_{\mathcal{O}} \Omega \to \mathcal{M}_K$ contains the generic point of $\mathcal{M}_K$. Thus, if a global section $f \in H^0(\mathcal{M}_K, \omega_{\mathcal{M}_K}^{\otimes k})$ were to have vanishing q-expansion, it must vanish on a dense open subset of $\mathcal{M}_K$. Since $\omega_{\mathcal{M}_K}^{\otimes k}$ is invertible over the integral affine scheme $\mathcal{M}_K$, its global sections inject into the stalk over the generic point, hence f must be the zero section.

□

Corollary 7.1.5 (q-expansion principle)**.** *Under the hypotheses of 7.1.4, let K be an $\mathcal{O}$-module, $L \subset K$ a submodule. Let $f \in H^0(\mathcal{M}, K \otimes \omega^{\otimes k})$ be such that its q-expansion lies in the submodule $L \otimes_{\mathcal{O}} \Omega \subset K \otimes_{\mathcal{O}} \Omega$, then f comes from an element of $H^0(\mathcal{M}, L \otimes \omega^{\otimes k})$.*

Proof. The exact sequence $0 \to L \to K \to K/L \to 0$ of $\mathcal{O}$-modules gives an exact sequence of sheaves

$$0 \to L \otimes \omega^{\otimes k} \to K \otimes \omega^{\otimes k} \to (K/L) \otimes \omega^{\otimes k} \to 0$$

and hence a exact sequence of cohomology

$$0 \to H^0(\mathcal{M}, L \otimes \omega^{\otimes k}) \to H^0(\mathcal{M}, K \otimes \omega^{\otimes k}) \to H^0(\mathcal{M}, (K/L) \otimes \omega^{\otimes k})$$

Applying 7.1.4 to the image of f in $H^0(\mathcal{M}, (K/L) \otimes \omega^{\otimes k})$, we find that the image is 0, and hence f comes from $H^0(\mathcal{M}, L \otimes \omega^{\otimes k})$. □

Lemma 7.1.6. *For any ring A, the map $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} A \to A(\!(q)\!)$ is injective.*

Proof. This follows from Lemma 2.6 in Conrad's notes *Algebraic theory of q-expansions.* □

I don't have a useful classification of the $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebras Ω which satisfy property $(\star)$ of 7.1.4, though the property seems to be satisfied by all rings one might consider in practice. For example, we have:

Proposition 7.1.7. *Let $\mathcal{O}$ be a Dedekind domain, then a $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra Ω will satisfy condition $(\star)$ of 7.1.4 in the following cases:*

(1) $\Omega = \mathcal{O}(\!(q)\!)$

(2) $\Omega = \mathcal{O}'(\!(q)\!)$ *for any finite flat extension $\mathcal{O}'$ of $\mathcal{O}$.*

(3) If Ω' satisfies $(\star)$, then for any morphism of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebras $\Omega \to \Omega'$, Ω will also satisfy $(\star)$ (for example, if $\Omega \subset \Omega'$ is a subalgebra).

(4) If Ω' satisfies $(\star)$, and we have an injection of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebras $\Omega' \subset \Omega$ with the $\mathcal{O}$-module quotient Ω/Ω' torsion-free (equivalently, flat), then Ω will satisfy $(\star)$.

Proof. The morphism in $(\star)$ is given by the homomorphism of $\mathcal{O}$-algebras

$$\mathcal{O}[j] \to \mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O} \to \Omega$$

where the first map sends j to its q-expansion $j(q) = q^{-1} + 744 + O(q)$ (and is clearly injective). We wish to show that the composition remains injective upon tensoring with K where $K = \operatorname{Frac} \mathcal{O}$ or $K = \mathcal{O}/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathcal{O}$. In case (1), where $\Omega = \mathcal{O}(\!(q)\!)$, for $K = \operatorname{Frac} \mathcal{O}$, then the injectivity follows from the $\mathcal{O}$-flatness of K and the injectivity of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O} \to \mathcal{O}(\!(q)\!)$ (7.1.6). Thus, let $K = \mathcal{O}/\mathfrak{m}$. In this case, since $\mathcal{O}/\mathfrak{m}$ is $\mathcal{O}$-finite, $\mathcal{O}(\!(q)\!) \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m}) = (\mathcal{O}/\mathfrak{m})(\!(q)\!)$, so we must demonstrate the injectivity of the map

$$(\mathcal{O}/\mathfrak{m})[j] \to (\mathcal{O}/\mathfrak{m})(\!(q)\!)$$

Indeed, if $c_n j^n + \cdots + c_1 j + c_0 \in (\mathcal{O}/\mathfrak{m})[j]$ is a nonzero polynomial of minimium degree with zero image in $(\mathcal{O}/\mathfrak{m})(\!(q)\!)$, then we may assume $c_n \neq 0 \in (\mathcal{O}/\mathfrak{m})$. But then its image looks like $c_n q^{-n} + O(q^{-n+1})$, which can only be 0 if $c_n = 0$, a contradiction.

In case (2), we wish to demonstrate that the composition

$$\mathcal{O}[j] \to \mathcal{O}(\!(q)\!) \to \mathcal{O}'(\!(q)\!)$$

remains injective after applying $\otimes_{\mathcal{O}}(\mathcal{O}/\mathfrak{m})$. We have already demonstrated this for the first map, so we wish to show that $\mathcal{O}(\!(q)\!) \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m}) \to \mathcal{O}'(\!(q)\!) \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$ is always injective. For this, it suffices to show that $\mathrm{Tor}_1^{\mathcal{O}}(*, \mathcal{O}'(\!(q)\!)/\mathcal{O}(\!(q)\!)) = 0$, or equivalently that the $\mathcal{O}$-module quotient $\mathcal{O}'(\!(q)\!)/\mathcal{O}(\!(q)\!)$ is flat (equivalently torsion-free). To demonstrate torsion-freeness, it suffices to show that if $x \in \mathcal{O}'$ and $a \in \mathcal{O}$ such that $ax \in \mathcal{O}$, then $x \in \mathcal{O}$. In this case, let $L := \operatorname{Frac} \mathcal{O}$, then we have a commutative diagram with all morphisms injective

$$\begin{array}{ccc} \mathcal{O}' \otimes_{\mathcal{O}} L & \longleftarrow & \mathcal{O}' \\ \uparrow & & \uparrow \\ L & \longleftarrow & \mathcal{O} \end{array}$$

Since $ax \in \mathcal{O}$, this implies that $a^{-1}ax = x \in L$, but since $x \in \mathcal{O}'$, x is integral over $\mathcal{O}$, but $\mathcal{O}$ is integrally closed, so this implies that $x \in \mathcal{O}$. This proves (2), and the exact same argument establishes (4).

For (3), if $\mathcal{O}[j] \to \Omega \to \Omega'$ is injective on all fibers over $\mathcal{O}$, then certainly the same must be true of $\mathcal{O}[j] \to \Omega$. $\square$

7.2 Bounded denominators

Remark 7.2.1. In order to connect the notions of analytic q-expansions of modular forms, and arithmetic q-expansions at level structures on $\mathrm{Tate}(q)/\Omega$, one must base change to $\mathbb{C}$. The resulting statement one gets might look somewhat strange. We consider some examples.

Example 7.2.2. Under the hypotheses of 7.1.4, suppose furthermore that $\mathcal{O} \subset \mathbb{C}$ is a subring, $\mathcal{M}_{\mathbb{C}}$ is connected, and that there is a map $h : \Omega \to \mathbb{C}(\!(q^{1/n})\!)$ of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebras for some integer $n \geq 1$. Then, we get a commutative diagram with the middle square cartesian:

$$\begin{array}{ccccc} \operatorname{Spec} \mathbb{C}(\!(q^{1/n})\!) & \longrightarrow & \operatorname{Spec} \mathbb{C} \otimes_{\mathcal{O}} \Omega & \xrightarrow{(\mathrm{Tate}(q)/\mathbb{C}\otimes\Omega,\alpha)} & \mathcal{M}_{\mathbb{C}} \\ & \searrow^{\operatorname{Spec}(h)} & \downarrow & & \downarrow \\ & & \operatorname{Spec} \Omega & \xrightarrow{(\mathrm{Tate}(q)/\Omega,\alpha)} & \mathcal{M} \end{array} \tag{18}$$

The first map on the top row is given by the $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra map $h : \Omega \to \mathbb{C}(\!(q^{1/n})\!)$ and the canonical inclusion $\mathbb{C} \hookrightarrow \mathbb{C}(\!(q^{1/n})\!)$, and by commutativity, the composite of the top row is given by $(\mathrm{Tate}(q)/\mathbb{C}(\!(q^{1/n})\!), \alpha)$. If $\mathcal{M}_{\mathbb{C}}^{\mathrm{an}} \cong [\mathcal{H}/\Gamma]$, then the process of q-expansion (c.f. 7.1.3) by pulling back an element of $H^0(\mathcal{M}_{\mathbb{C}}, \omega_{\mathcal{M}_{\mathbb{C}}}^{\otimes k}) = H^0(\mathcal{M}, \mathbb{C} \otimes_{\mathcal{O}} \omega_{\mathcal{M}}^{\otimes k})$ and dividing by ω_{can} actually gives you a q-expansion defined analytically of the corresponding element of $M_k(\Gamma)$. As a consequence, noting that $\mathbb{C} \otimes_{\mathcal{O}} \Omega$ consists of *finite* $\mathbb{C}$-linear combinations of elements of Ω, we can sometimes obtain nontrivial statements about the Fourier expansions of weakly holomorphic modular forms (c.f. 7.2.4).

Note that if $\mathcal{M}$ is connected, but not geometrically connected (ie $\mathcal{M}_{\mathbb{C}}$ is not connected), then by 7.1.4, the q-expansion map $H^0(\mathcal{M}_{\mathbb{C}}, \omega_{\mathcal{M}_{\mathbb{C}}}^{\otimes k}) \to \mathbb{C} \otimes_{\mathcal{O}} \Omega$ associated to the map $\operatorname{Spec} \mathbb{C} \otimes_{\mathcal{O}} \Omega \to \mathcal{M}_{\mathbb{C}}$ will still be injective, but its composition with $\mathbb{C} \otimes_{\mathcal{O}} \Omega \to \mathbb{C}(\!(q^{1/n})\!)$ will not be injective. Indeed, supposing for simplicity that $\mathcal{O} \subset \mathbb{C}$ is a *subfield*, let M be the coarse scheme of $\mathcal{M}$, then since the map $\operatorname{Spec} \Omega \to \mathcal{M} \to M$ is assumed to contain the generic point, since M is not geometrically connected, by [Sta16] 04KV, it must be the case that $\mathcal{O}$ is not algebraically closed in $\operatorname{Frac} \Omega$. This implies that $\mathbb{C} \otimes_{\mathcal{O}} \Omega$ will be a product of extensions of Ω, and the q-expansion map induced by $\operatorname{Spec} \mathbb{C} \otimes_{\mathcal{O}} \Omega \to \mathcal{M}_{\mathbb{C}}$ will actually be the product of the q-expansion maps at a cusp on each component of $\mathcal{M}_{\mathbb{C}}$. Its composition with $\mathbb{C} \otimes_{\mathcal{O}} \Omega \to \mathbb{C}(\!(q^{1/n})\!)$ will correspond to the q-expansion map at a particular cusp, corresponding to the choice of the $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebra morphism $h : \Omega \to \mathbb{C}(\!(q^{1/n})\!)$.

Lemma 7.2.3. *Let $\mathcal{O} \subset \overline{\mathbb{Q}}$ be a subring. Then we have an equality of subrings of $\mathbb{C}(\!(q)\!)$:*

$$\Big(\mathcal{O}(\!(q)\!) \otimes_{\mathcal{O}} \mathbb{C}\Big) \cap \overline{\mathbb{Q}}(\!(q)\!) = \mathcal{O}(\!(q)\!) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}$$

Proof. It's clear that the right side is contained in the left side. Let L be either $\overline{\mathbb{Q}}$ or $\mathbb{C}$. Given $f = \sum_i b_i q^i \in \overline{\mathbb{Q}}(\!(q)\!)$, f is in $\mathcal{O}(\!(q)\!) \otimes_{\mathcal{O}} L$ if and only if there exist finitely many $c_1, \ldots, c_n \in L$ and $g_j = \sum_i a_{ji} q^i \in \mathcal{O}(\!(q)\!)$ for

$j \in \{1, \dots, n\}$ such that $f = \sum_{j=1}^n c_j g_j$, or equivalently such that $b_i = \sum_{j=1}^n c_j a_{ji}$ for all $i \in \mathbb{Z}$. Given any choice of $g_j \in \mathcal{O}(\!(q)\!)$, the existence of the c_j's amounts to solving a system of linear equations in finitely many variables, and hence if a solution exists in any extension of $\overline{\mathbb{Q}}$, it must exist in $\overline{\mathbb{Q}}$ itself. This implies our desired equality. Of course this all holds with $\overline{\mathbb{Q}}, \mathbb{C}$ replaced by any extension of fields containing $\mathcal{O}$. □

Example 7.2.4. Let p be a rational prime, and A the localization of the ring of integers of some number field at a prime lying over p. Let $\mathcal{M} \to \mathcal{M}(1)_A$ be a finite etale morphism with $\mathcal{M}_{\mathbb{C}}$ connected with $\mathcal{M}_{\mathbb{C}}^{\text{an}} \cong [\mathcal{H}/\Gamma]$. From (15), the scheme of $\mathcal{M}$-level structures over $\text{Tate}(q)/A(\!(q)\!)$ is finite etale over $\operatorname{Spec} A(\!(q)\!)$, and hence by Corollary 5.4.3 of [?Chen17] it becomes completely decomposed over $\mathcal{O}(\!(q^{1/e})\!)$, where $\mathcal{O}$ is finite etale over A and e is coprime to p. In particular, p is not invertible in $\mathcal{O}$. Base changing $\mathcal{M} \to \mathcal{M}(1)_A$ to $\mathcal{O}$, choosing an embedding $\mathcal{O} \subset \mathbb{C}$, and setting $\Omega := \mathcal{O}(\!(q^{1/e})\!)$ with the natural map to $\mathbb{C}(\!(q^{1/e})\!)$, we find that the scheme of $\mathcal{M}$-level structures over $\text{Tate}(q)/\Omega$ is completely decomposed. Thus, from 7.2.2, we find that all Fourier expansions (ie, at all cusps) of all weakly holomorphic modular forms for Γ lie in $\mathbb{C} \otimes_{\mathcal{O}} \mathcal{O}(\!(q^{1/e})\!)$. It follows from the lemma that every modular form with algebraic Fourier coefficients must have Fourier expansions in $\mathcal{O}(\!(q^{1/e})\!) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}$. That is to say, they have bounded denominators at p.

Theorem 7.2.5 (Bounded denominators)**.** *Let $B \subset \overline{\mathbb{Q}}$ be a subring, let $\mathcal{M} \to \mathcal{M}(1)_B$ be finite etale morphism. Suppose $\mathcal{M}_{\mathbb{C}}$ is connected with analytification isomorphic to $[\mathcal{H}/\Gamma]$. Suppose p is not invertible in B, then any modular form $f \in M_k(\Gamma)$ with algebraic Fourier coefficients has bounded denominators at p.*

Proof. By considering a presentation of $\mathcal{M}$, we find that the map $\mathcal{M} \to \mathcal{M}(1)_B$ is the base change of a finite etale morphism $\mathcal{M}_A \to \mathcal{M}(1)_A$ via a map $\operatorname{Spec} B \to \operatorname{Spec} A$ with A a finite type $\mathbb{Z}$-algebra with p not invertible. Then, localizing at a prime lying over p, we are reduced to the situation of 7.2.4, which gives us bounded denominators at p. □

7.3 Arithmetic models

Situation 7.3.1. Let $\mathcal{O} \subset \mathbb{C}$ be a Dedekind subring such that there is a prime $p \in \mathcal{O}^\times$. Let $\mathcal{M} \to \mathcal{M}(1)_{\mathcal{O}}$ be finite etale with $\mathcal{M}_{\mathbb{C}}$ connected. Choose a uniformization $\mathcal{M}_{\mathbb{C}}^{\text{an}} \cong [\mathcal{H}/\Gamma]$ as in §5.3. Suppose there exists an $\mathcal{M}$-level structure α on $\text{Tate}(q)/\mathcal{O}(\!(q^{1/n})\!)$, and choose an embedding of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathcal{O}$-algebras $\mathcal{O}(\!(q^{1/n})\!) \hookrightarrow \mathbb{C}(\!(q^{1/n})\!)$. Then taking q-expansions at $(\text{Tate}(q)/\mathcal{O}(\!(q^{1/n})\!), \alpha)$, we obtain a q-expansion map

$$H^0(\mathcal{M}_{\mathbb{C}}, \omega_{\mathcal{M}_{\mathbb{C}}}^{\otimes k}) = H^0(\mathcal{M}, \mathbb{C} \otimes_{\mathcal{O}} \omega^{\otimes k}) \longrightarrow \mathbb{C} \otimes_{\mathcal{O}} \mathcal{O}(\!(q^{1/n})\!) \subset \mathbb{C}(\!(q^{1/n})\!)$$

From the discussion in 7.2.2, the composition of this map with the isomorphism $M_k(\Gamma) \cong H^0(\mathcal{M}_{\mathbb{C}}, \omega_{\mathcal{M}_{\mathbb{C}}}^{\otimes k})$ coming from our choice of uniformization is given by Fourier expansion at some cusp. By changing our choice of uniformization (which may involve changing Γ), we may assume that the resulting map $M_k(\Gamma) \to \mathbb{C}(\!(q^{1/n})\!)$ is given by taking expansions in the uniformizer $e^{2\pi i\tau/n}$ (ie, q-expansion at $i\infty$). On the other hand, by the q-expansion principle, we find that $H^0(\mathcal{M}, \omega^{\otimes k})$ is identified with the submodule of $H^0(\mathcal{M}, \mathbb{C} \otimes_{\mathcal{O}} \omega^{\otimes k}) \cong M_k(\Gamma)$ whose Fourier expansions lie in the subring $\mathcal{O}(\!(q^{1/n})\!) \subset \mathbb{C}(\!(q^{1/n})\!)$. In particular, this implies:

Theorem 7.3.2 (Arithmetic models)**.** *In situation 7.3.1, taking $k = 0$, let $M_0(\Gamma, \mathcal{O})$ denote the ring of modular functions for Γ whose $e^{2\pi i\tau/n}$-expansions lie in $\mathcal{O} \subset \mathbb{C}$. Let M be the coarse moduli scheme of $\mathcal{M}$, then $M \cong$* $\operatorname{Spec} M_0(\Gamma, \mathcal{O})$.

Proof. The discussion above proved that $H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) \cong M_0(\Gamma, \mathcal{O})$. It remains to show that $M \cong \operatorname{Spec} H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$. If $\mathcal{M}$ is representable, then $\mathcal{M} = M$ and the result is clear. In the general case, let $\mathcal{M}(p^2)$ denote the fine moduli scheme over $\mathcal{O}$ parametrizing elliptic curves with full level p^2-structures. Then $\mathcal{M}(p^2) \to \mathcal{M}(1)$ is finite etale (since $p \in \mathcal{O}^\times$) and G-Galois, where $G = \text{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$. Then, $\mathcal{M}' := \mathcal{M}(p^2) \times_{\mathcal{M}(1)} \mathcal{M}$ is an affine scheme, and the coarse scheme M can be identified with the quotient of $\mathcal{M}'$ by G in the category of schemes. Thus,

$$M = \operatorname{Spec} H^0(\mathcal{M}', \mathcal{O}_{\mathcal{M}'})^G,$$

but $\mathcal{M}' \to \mathcal{M}$ is a covering of the etale site $\mathcal{M}_{\acute{e}t}$, and hence by the sheaf condition, we have

$$H^0(\mathcal{M}', \mathcal{O}_{\mathcal{M}'})^G = H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$$

which proves the result. □

8 Appendix - Abstract deformation theory

Classically the deformation theory of an object provides a local picture of the associated moduli stack of the object. However, given an abstract stack which may or may not have a moduli interpretation, one can still define an "abstract" deformation theory for the objects of the stack, using which one can then apply Artin approximation to understand the relation between etale local rings and universal deformation rings of DM stacks (c.f. 8.0.6), as well as their relation with the etale local rings of their coarse moduli schemes (c.f. 8.0.7). What appears here are detailed statements of the relevant definitions and some results, with references to the stacks project where the proofs may be found. Hopefully this is more readable than reading the stacks project directly, where the relevant material is spread over 600+ pages.

Let $\mathcal{M}$ be an algebraic stack[20] over a scheme S. That is, we have a functor $p : \mathcal{M} \to \underline{\mathbf{Sch}}/S$ such that

- $p : \mathcal{M} \to \underline{\mathbf{Sch}}/S$ is *fibered in groupoids*,
- $p : \mathcal{M} \to \underline{\mathbf{Sch}}/S$ is a *stack*: I.e., the isom functors are sheaves for the etale topology on $\underline{\mathbf{Sch}}/S$, and any etale descent datum for objects of $\mathcal{X}$ is effective,
- $p : \mathcal{M} \to \underline{\mathbf{Sch}}/S$ is *algebraic*: I.e., the diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times_S \mathcal{M}$ is representable by algebraic spaces, and there is an S-scheme U, and a smooth surjective morphism $U \to \mathcal{M}$.

In the above, if $U \to \mathcal{M}$ can moreover be chosen to be *etale*, then $\mathcal{M}$ is called a Deligne-Mumford (DM) stack. This is true if and only if the diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times_S \mathcal{M}$ is unramified.

Moreover, we will assume that

- S is locally Noetherian and $p : \mathcal{M} \to \underline{\mathbf{Sch}}/S$ is *locally of finite type*: Ie the scheme U can be chosen to be locally of finite type over S.

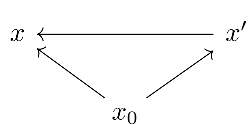
Let k be a field, and $i : \operatorname{Spec} k \to S$ a morphism of finite type. This amounts to saying that if $s \in S$ is the image of i, then i induces a finite extension of the residue field $k \supset \kappa(s)$. Moreover, i factors through the inclusion of an affine open $\operatorname{Spec}\Lambda \subset S$ such that induced map $\Lambda \to k$ is finite, making k into a Λ-algebra. The fact that S is locally Noetherian forces Λ to be Noetherian. We now define the category $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$ as follows [Sta16, 06GB]

- The objects of $\mathcal{C}_\Lambda$ are pairs (A, φ) where A is an Artinian local Λ-algebra and $\varphi : A/\mathfrak{m}_A \to k$ is an isomorphism of Λ-algebras.
- A morphism $(B, \psi) \to (A, \varphi)$ in $\mathcal{C}_\Lambda$ is given by a local Λ-algebra homomorphism $f : B \to A$ such that if $\overline{f}$ denotes the induced map of residue fields, then $\varphi \circ \overline{f} = \psi$.

One can check that $\mathcal{C}_\Lambda$ is equivalent to the opposite of the category of factorizations $\operatorname{Spec} k \to \operatorname{Spec} A \to S$ of i such that A is Artinian local and the induced map $A \to k$ identifies k with the residue field of A.

Let $x_0 : \operatorname{Spec} k \to \mathcal{M}$ be a morphism corresponding to an object of the fiber category $\mathcal{M}(\operatorname{Spec} k \to S)$, which we also call x_0. We define the category $\mathcal{F}_{x_0} := \mathcal{F}_{\mathcal{M},k,x_0}$ as follows [Sta16, 07T2]:

- Its objects are morphisms $x_0 \to x$ in $\mathcal{M}$ where $p(x) = \operatorname{Spec} A$ with A an Artinian local ring and $\operatorname{Spec} k = p(x_0) \to p(x) \to S$ is a factorization of $i : \operatorname{Spec} k \to S$ inducing an isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} k$.
- A morphism $(x_0 \to x) \to (x_0 \to x')$ is a commutative diagram in $\mathcal{M}$



Note that the arrows are reversed in the definition of a morphism. There is a natural map, which we also call $p : \mathcal{F}_{x_0} \to \mathcal{C}_\Lambda$ sending $(x_0 \to x) \mapsto A$, where $\operatorname{Spec} A = p(x)$.

[20]see [Sta16] Tag 026O and [AV02] §2

Theorem 8.0.1. *The functor $p : \mathcal{F}_{x_0} \to \mathcal{C}_\Lambda$ defined above is a "deformation category".*

Proof. A predeformation category over $\mathcal{C}_\Lambda$ is by definition a category fibered in groupoids over $\mathcal{C}_\Lambda$ such that the fiber category over k is equivalent to a category with a single object and a single morphism [Sta16, 06GS]. Thus $p : \mathcal{F}_{x_0} \to \mathcal{C}_\Lambda$ is visibly a predeformation category. A deformation category is a predeformation category which satisfies the Rim-Schlessinger (RS) conditions [Sta16, 06J1]. By [Sta16, 07WU], if $\mathcal{M}$ is a category fibered in groupoids satisfying (RS), then the predeformation category $\mathcal{F}_{x_0}$ is a *deformation category.* By [Sta16, 07WQ], any algebraic stack over a locally Noetherian scheme S satisfies (RS). □

Remark 8.0.2. For any $A \in \mathcal{C}_\Lambda$, in the classical situation the objects $x_0 \to x$ of $\mathcal{F}_{x_0}(A)$ are just deformations of x_0 over A, and the automorphisms of $x_0 \to x$ of $\mathcal{F}_{x_0}(A)$ are precisely the automorphisms of x in $\mathcal{M}(A)$ which restrict to the identity on x_0. Namely, these are "infinitesimal automorphisms" [Sta16, 06JN].

Since $\mathcal{F}_{x_0}$ is cofibered in groupoids, the sets of isomorphism classes of objects in its fiber categories define a functor[21] $\overline{\mathcal{F}}_{x_0} : \mathcal{C}_\Lambda \to \underline{\textbf{Sets}}$ sending $A \in \mathcal{C}_\Lambda$ to the set of isomorphism classes $\pi_0(\mathcal{F}_{x_0}(A))$. We wish to show that this functor is pro-representable if $\mathcal{M}$ is DM. For this, we use Schlessinger's conditions:

Theorem 8.0.3. *$\overline{\mathcal{F}}_{x_0}$ is prorepresentable if and only if the following are satisfied*

(a) $\overline{\mathcal{F}}_{x_0}$ is a deformation functor (i.e. its associated category over $\mathcal{C}_\Lambda$ is a deformation category)

(b) $\dim_k T\overline{\mathcal{F}}_{x_0}$ is finite, and

(c) $\gamma : \mathrm{Der}_\Lambda(k,k) \to T\overline{\mathcal{F}}_{x_0}$ is injective.

Moreover, condition (a) is equivalent to the condition:

(a') For every morphism $x' \to x$ in $\mathcal{F}_{x_0}$ lying over a surjection $A' \to A$ in $\mathcal{C}_\Lambda$, the map $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$ is surjective.

Proof. The criteria for prorepresentability is [Sta16, 06JM]. To see that (a) is equivalent to (a'), note that $\overline{\mathcal{F}}_{x_0}$ is visibly a predeformation functor, so it is a deformation functor if and only if it satisfies (RS). Various equivalent conditions to $\overline{\mathcal{F}}_{x_0}$ satisfying (RS) are given in [Sta16, 06J8], one of which is (a'). □

Remark 8.0.4. In the classical case the condition (a') amounts to saying that every automorphism of a deformation extends to higher order extensions of the deformation.

Theorem 8.0.5. *If $\mathcal{M}$ is a Deligne-Mumford stack locally of finite type over a locally noetherian scheme S, and assume the factorization $\mathrm{Spec}\, k \to \Lambda$ discussed above induces a separable extension of residue fields, then $\overline{\mathcal{F}}_{x_0}$ is pro-representable.*

Proof. We will verify the conditions (a'), (b), and (c) of Theorem 8.0.3. The separability assumption implies that $\Omega_{k/\Lambda} = 0$, so (c) holds trivially, since $\mathrm{Der}_\Lambda(k,k) = \mathrm{Hom}_k(\Omega_{k/\Lambda}, k) = 0$. Condition (b) is a consequence of $\mathcal{M}$ being locally of finite type [Sta16, 07X1]. To check (a') by Remark 8.0.2, it would suffice to check that $\mathrm{Aut}_A(x)$ is trivial for every object $x \in \mathcal{F}_{x_0}$ (i.e., there are no infinitesimal automorphisms). Since $\mathcal{M}$ is assumed Deligne-Mumford, the diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times_S \mathcal{M}$ is unramified (in particular, formally unramified), so its inertia stack is also formally unramified, which is precisely to say that there are no infinitesimal automorphisms. □

Lemma 8.0.6. *Let $\mathcal{M}$ be an algebraic stack locally of finite type over a locally Noetherian scheme S. Let k be a field and $x : \mathrm{Spec}\, k \to \mathcal{M}$ be a morphism such that $\mathrm{Spec}\, k \to S$ is finite type with image $s \in S$. There exists a versal ring R to $\mathcal{M}$ at x. If $\mathcal{O}_{S,s}$ is a G-ring*[22]*, then we may find an smooth morphism $U \to \mathcal{M}$ with U a finite type S-scheme, and a point $u \in U$ with residue field k, such that*

(1) $\mathrm{Spec}\, k \to U \to \mathcal{M}$ coincides with the given morphism x,

(2) there is an isomorphism $\widehat{\mathcal{O}_{U,u}} \cong R$.

[21] In the classical situation this is effectively the *deformation functor* of the object x_0. However we avoid using that terminology in this abstract setting since "deformation functor" has a technical meaning.

[22] basically everything that arises in practice is a G-ring

If $\mathcal{M}$ is moreover Deligne-Mumford, then if we choose R to be a universal deformation ring (8.0.5), then the morphism $U \to \mathcal{M}$ can be moveover chosen to be etale.

Proof. Everything but the final statement is just [Sta16] Tag 0DR0. For the last part, V be a etale cover of $\mathcal{M}$, and let $v \in V$ be a point lying over x, so that $\kappa(v)/k$ is a finite etale extension. We wish to show that $f : U \times_{\mathcal{M}} V \to V$ is etale at (v, u). Since f is smooth, it suffices to show that it is (locally) quasi-finite at (v, u). Let $f_v : U_v \to \operatorname{Spec} \kappa(v)$ be the fiber of f above v, and let u_v denote the point of U_v lying over u. By [Sta16] Tag 01TH, it suffices to show that u_v is closed in U_v and there does not exist a point $\eta_v \in U_v$ which specializes to u_v. For the first part, u_v is clearly closed in U_v because $U_v \to \operatorname{Spec} \kappa(v)$ is separated, and u_v is a section.

For the second part, note that if $\eta_v \in U_v$ specializes to u_v, then its image $\eta \in U$ must specialize to u. Thus, we must show that any $\eta \in U$ specializing to u does not lie in the fiber U_v. Suppose $\eta \in U$ specializes to $u \in U$. Then, since completions are faithfully flat, η lifts to a point in $\widehat{\mathcal{O}_{U,u}}$ corresponding to some non-maximal prime ideal $\mathfrak{p}$ with residue field $\kappa(\eta) = (\widehat{\mathcal{O}_{U,u}})_{\mathfrak{p}}/\mathfrak{p}$, which is an Artinian local Λ-algebra. The map $g : \operatorname{Spec} \kappa(\eta) \to \operatorname{Spec} \widehat{\mathcal{O}_{U,u}}$ corresponds to an object in the deformation category $\mathcal{F}_{\mathcal{M},k,x}$ over $\operatorname{Spec} \kappa(\eta)$. Clearly g does not factor through $\operatorname{Spec} \kappa(v)$. Thus, since $\widehat{\mathcal{O}_{U,u}}$ is universal, the object of $\mathcal{F}_{\mathcal{M},k,x}$ corresponding to g is not isomorphic to the pullback of some object of $\mathcal{M}(\operatorname{Spec} \kappa(v))$. By the definition of the 2-fiber product U_v, this means that there do not exist any points of U_v lying over η.

□

Proposition 8.0.7. *Let $\mathcal{M}$ be a smooth 1-dimensional Deligne-Mumford stack over $\mathbb{C}$, and let $c : \mathcal{M} \to M$ be the canonical map to its coarse moduli scheme M. Let $x \in \mathcal{M}$ be a geometric point with image $\overline{x} \in M$. Let $\mathcal{O}_{\mathcal{M},x}$ be the etale local ring at x, and $\mathcal{O}_{M,\overline{x}}$ the etale local ring at $\overline{x}$. By Lemma 8.0.6, $\widehat{\mathcal{O}_{\mathcal{M},x}}$ can be identified with the universal deformation ring of x. Let $G_x := \operatorname{Aut}_{\mathcal{M}}(x)$, and let $K_x \subset G_x$ the subgroup of automorphisms which extend to the universal deformation of x over $\widehat{\mathcal{O}_{\mathcal{M},x}}$. Then the map $c_x : \operatorname{Spec} \mathcal{O}_{\mathcal{M},x} \to \operatorname{Spec} \mathcal{O}_{M,\overline{x}}$ is a finite flat totally ramified extension of DVR's with ramification index equal to the order of the group G_x/K_x.*

Proof. Let $\mathcal{M}_{(x)} := \mathcal{M} \times_M \operatorname{Spec} \mathcal{O}_{M,\overline{x}}$. From the proof of Theorem 11.3.1 of [Ols16], we find that

$$\mathcal{M}_{(x)} \cong [\operatorname{Spec} \mathcal{O}_{\mathcal{M},x}/G_x]$$

and moreover the composition

$$\operatorname{Spec} \mathcal{O}_{\mathcal{M},x} \to [\operatorname{Spec} \mathcal{O}_{\mathcal{M},x}/G_x] \to \operatorname{Spec} \mathcal{O}_{M,\overline{x}}$$

is finite. Since the map $\operatorname{Spec} \mathcal{O}_{M,\overline{x}} \to M$ is flat, the projection $[\operatorname{Spec} \mathcal{O}_{\mathcal{M},x}/G_x] \to \operatorname{Spec} \mathcal{O}_{M,\overline{x}}$ identifes the target with the coarse moduli scheme of $[\operatorname{Spec} \mathcal{O}_{\mathcal{M},x}/G_x]$ (Theorem 11.1.2 of the same book). Thus, we have $\mathcal{O}_{M,\overline{x}} = (\mathcal{O}_{\mathcal{M},x})^{G_x}$, and since $\mathcal{O}_{\mathcal{M},x}$ is a DVR with residue characteristic 0, $\mathcal{O}_{M,\overline{x}}$ is also a DVR. In particular, the map $\operatorname{Spec} \mathcal{O}_{\mathcal{M},x} \to \operatorname{Spec} \mathcal{O}_{M,\overline{x}}$ is a finite map between regular schemes, and hence is flat. Since they are also strictly henselian, the map is totally ramified. Let K be the kernel of the action of G_x on $\mathcal{O}_{\mathcal{M},x}$, then the ramification index is $|G_x/K|$. We wish to show that $K = K_x$.

To see this, we use the fact that by 8.0.6, the completion $\widehat{\mathcal{O}_{\mathcal{M},x}}$ is the universal deformation ring of the object x. Thus, it represents the functor F_x, which to every Artinian local $\mathbb{C}$-algebra A associates the set of isomorphism classes:

$$F_x(A) = \{(X/A, \varphi : X_0 \xrightarrow{\sim} x\}/\cong$$

where X_0 is the special fiber of X/A. One can check that the action of G_x on $\widehat{\mathcal{O}_{\mathcal{M},x}}$ induces the following action on the functor F_x:

$$g \cdot (X/A, \varphi) := (X/A, g \circ \varphi) \qquad g \in G_x$$

Thus, the kernel of the action consists precisely of $g \in G_x$ which extend to every deformation of x. Since $\mathcal{M}$ is Deligne-Mumford, any such extension is unique, and hence K is equivalently the set of automorphisms which extend to the universal deformation of x over $\widehat{\mathcal{O}_{\mathcal{M},x}}$, which is what we wanted to show.

□

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