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Nonabelian level structures, Nielsen equivalence, and Markoff triples

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Abstract

In this paper we establish a congruence on the degree of the map from a component of a Hurwitz space of covers of elliptic curves to the moduli stack of elliptic curves. Combinatorially, this can be expressed as a congruence on the cardinalities of Nielsen equivalence classes of generating pairs of finite groups. Building on the work of Bourgain, Gamburd, and Sarnak, we apply this congruence to show that for all but finitely many primes p, the group of Markoff automorphisms acts transitively on the non-zero $\mathbb{F}_p$-points of the Markoff equation $x^2 + y^2 + z^2 - 3xyz = 0$. This yields a strong approximation property for the Markoff equation, the finiteness of congruence conditions satisfied by Markoff numbers, and the connectivity of a certain infinite family of Hurwitz spaces of $\mathrm{SL}_2(\mathbb{F}_p)$-covers of elliptic curves. With possibly finitely many exceptions, this resolves a conjecture of Bourgain, Gamburd, and Sarnak, first posed by Baragar in 1991, and a question of Frobenius, posed in 1913. Since their methods are effective, this reduces the conjecture to a finite computation.

Contents

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1. **Introduction**

Let k be a field. Let $\mathcal{M}_{g,n}$ denote the moduli stack over k of smooth curves of genus g with n marked points. For an integer d, there is a Hurwitz stack $\mathcal{H}_{g,n,d}$ which classifies connected degree d covers of smooth (g,n)-curves, ramified above the n marked points and unramified elsewhere. While $\mathcal{M}_{g,n}$ is irreducible, the Hurwitz stacks $\mathcal{H}_{g,n,d}$ are typically disconnected. It is a classical problem to find combinatorial invariants of covers that can distinguish their connected components, or equivalently, to determine whether an open and closed substack of $\mathcal{H}_{g,n,d}$ corresponding to some fixed values of various combinatorial invariants is connected.

For example, when $k = \mathbb{C}$, Clebsch, Lüroth, and Hurwitz used a combinatorial argument [Cle73], [Hur91], [Ful69] to show that the substack of $\mathcal{H}_{0,n,d}$ corresponding to covers which are simply branched over the n marked points is either empty or connected, the latter happening exactly when n is even and $n \geq 2d-2$; taking d large enough so that every curve of genus g admits a degree d map to $\mathbb{P}^1$ with simple branching over n points, this gave the first proof of the connectedness of $\mathcal{M}_g$. When $(g,n) = (0,3)$ and $k = \mathbb{Q}$, the connected components of $\bigsqcup_{d\geq 1} \mathcal{H}_{0,3,d}$ are in bijection with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits of dessins d'enfants. Describing these orbits in terms of combinatorial data remains a deep and very open problem [HS97], [Sch97], [SL97a], [SL97b].

When a combinatorial type is fixed and g or n are allowed to be large, there are a number of connectedness results which go under the name of *genus stabilization* ($g \gg 0$ — see [DT06, §6] or [CLP16]), or *branch stabilization* ($g = 0$, $n \gg 0$), which first appeared as a theorem of Conway and Parker.[1] Recently such connectedness results have found notable applications to number theory. In [EVW16] and [LZB19], Ellenberg, Venkatesh, Westerland, Liu, Wood, and Zureick-Brown proved branch stabilization results and used them to study the function field analogs of Cohen-Lenstra-Martinet heuristics on class groups. In [RV15], Roberts and Venkatesh also used branch stabilization to study an open problem of Malle and Roberts on the infinitude of number fields with alternating or symmetric monodromy group and bounded ramification.

When (g,n) are fixed and the combinatorial type is allowed to vary, much less is known, and it is unclear what one should even expect. In this paper, we will prove a connectedness result of this type, where $(g,n) = (1,1)$ (see Theorem 1.2.10), and apply it to a conjecture of Bourgain, Gamburd, and

[1] see [FV91, Appendix]; also see [Lön20], [EVW16, Th. 6.1], [LZB19, Cor. 12.5].

Sarnak on the Diophantine properties of the Markoff equation (see Section 1.2). The key new ingredient is a congruence on the degree of a forgetful map between moduli spaces; see Theorem 1.1.1. Our results also have applications to a non-congruence Rademacher's conjecture, an old question of Frobenius, and can also be interpreted in the language of Nielsen equivalence for generating pairs of finite groups.

1.1. *Admissible covers and the main congruence*. Let $\mathcal{M}(1) := \mathcal{M}_{1,1}$ be the moduli stack of elliptic curves, and let $\overline{\mathcal{M}(1)}$ denote its compactification by stable pointed curves of genus 1. Henceforth, following [DR73], the objects of $\overline{\mathcal{M}(1)}$ will be called 1-*generalized elliptic curves*. For a finite group G, let $\mathcal{A}dm(G)$ denote the stack of admissible G-covers of 1-generalized elliptic curves E which are unramified away from the marked point $O \in E$ (the *origin* of E).[2] The stacks $\mathcal{A}dm(G)$ are smooth, proper, and Deligne-Mumford over $\mathbb{Z}[1/|G|]$, but for the purposes of the introduction it will suffice to work over $\mathbb{C}$. The coarse schemes of $\mathcal{A}dm(G)$ and $\overline{\mathcal{M}(1)}$ are denoted $Adm(G)$ and $\overline{M(1)}$ respectively.

Let $\mathfrak{f} : \mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ be the forgetful map sending an admissible cover of E to the base curve E. To a cover $C \to E$ in $\mathcal{A}dm(G)$, one can associate the *Higman invariant*, which analytically is the conjugacy class in G given by monodromy around the origin $O \in E$. If $\mathfrak{c}$ is a conjugacy class of G, the substack $\mathcal{A}dm(G)_{\mathfrak{c}} \subset \mathcal{A}dm(G)$ classifying covers with Higman invariant $\mathfrak{c}$ is open and closed, and for any such cover $C \to E$, the ramification index of any point above $O \in E$ is equal to the order of any representative of $\mathfrak{c}$. In this paper we will establish, for any connected component $\mathcal{X} \subset \mathcal{A}dm(G)_{\mathfrak{c}}$ with coarse scheme X, a divisibility theorem on the degree of the finite flat forgetful map $\mathfrak{f} : X \to \overline{M(1)}$. For example, we will show

THEOREM 1.1.1 (See Corollary 4.12.5). *Let G be a finite group. Let $c \in G$, and let $\mathfrak{c}$ be its conjugacy class. Let $\mathcal{X} \subset \mathcal{A}dm(G)_{\mathfrak{c}}$ be a connected component with coarse scheme X, parametrizing covers with Higman invariant $\mathfrak{c}$. For a prime ℓ, write $r := \mathrm{ord}_\ell(|c|)$. For $x \in \mathbb{R}$, write $\lceil x \rceil$ for the minimum integer $\geq x$. Then we have*

(a) *Write $\mathrm{ord}_\ell(|G|) = r + s$. Let $j \geq 0$ be an integer such that G does not contain any proper normal subgroup of order divisible by ℓ^{j+1}. If $r \geq 3s + j$, then*
$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \ell^{\lceil \frac{r-3s-j}{2} \rceil}.$$
(b) *Suppose G is non-abelian and simple. If $\ell^{r+1} \geq |G|^{1/3}$ and G is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ for any q, then*
$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \ell^{\lceil \frac{r}{2} \rceil}.$$

[2]If E is smooth, these are just G-Galois branched covers, only branched above the origin. If E is nodal, we also allow non-étaleness above the node, though these are not considered ramification points; see Section 2.1.

A sketch of the proof is given in Section 1.3.2. A more precise version of this result is given in Theorems 3.5.1 and 4.10.5. Note that in case (a), if $\mathrm{ord}_\ell(|c|) = \mathrm{ord}_\ell(|G|)$ and G does not contain any proper normal subgroup of order divisible by ℓ, then we obtain a congruence $\equiv 0 \mod \ell^{\lceil \frac{r}{2} \rceil}$. This is the case when $c = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{F}_\ell)$ for $\ell \geq 3$, so we get a congruence $\equiv 0 \bmod \ell$. This case leads to our main result on Markoff triples (see Section 1.2.1).

Let $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ denote the substack classifying smooth covers. The forgetful map $\mathfrak{f} : \mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ restricts to an étale map $\mathcal{A}dm^0(G) \to \mathcal{M}(1)$, which in turn factors as

$$\mathcal{A}dm^0(G) \to \mathcal{M}(G) \to \mathcal{M}(1),$$

where $\mathcal{M}(G)$ is the moduli stack of elliptic curves with G-structures (see Section 2.5 below). Here the first map is a homeomorphism and the second map is finite étale. In particular, the sets of connected components of $\mathcal{A}dm(G), \mathcal{A}dm^0(G)$, and $\mathcal{M}(G)$ are in natural bijection with each other and can be characterized combinatorially using Galois theory: if Π denotes the fundamental group of a once punctured torus, then the geometric fibers of $\mathcal{M}(G)$ above $\mathcal{M}(1)$ are in bijection with the set

$$\mathrm{Epi}^{\mathrm{ext}}(\Pi, G) := \mathrm{Epi}(\Pi, G)/\,\mathrm{Inn}(G).$$

Note that Π is free of rank 2 and that $\Pi \to \Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ induces an *isomorphism* $\mathrm{Out}(\Pi) := \mathrm{Aut}(\Pi)/\,\mathrm{Inn}(\Pi) \cong \mathrm{GL}_2(\mathbb{Z})$ [LS01, Prop. 4.5]. Let $\mathrm{Out}^+(\Pi) \cong \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of $\mathrm{Out}(\Pi)$ acting with determinant 1 on $\mathbb{Z}^2$. Then the fundamental group of $\mathcal{M}(1)$ is the profinite completion of $\mathrm{Out}^+(\Pi)$, the connected components of $\mathcal{M}(G)$ are in bijection with the orbits of the natural action of $\mathrm{Out}^+(\Pi)$ on $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$, and the degree of the induced map on coarse schemes $Adm^0(G) \to M(1)$ is either equal to the cardinality of the orbit, or twice the cardinality (Theorem 2.5.2). From this we deduce the following combinatorial interpretation of Theorem 1.1.1:

COROLLARY 1.1.2. *Let G be a finite group. Let $c \in G$, and let $\mathfrak{c}$ be its conjugacy class. Fix a basis (a, b) for Π. Let $\varphi : \Pi \to G$ be a surjection satisfying $\varphi([a, b]) \in \mathfrak{c}$. Let $\mathrm{Out}^+(\Pi) \cdot \varphi$ denote the orbit of φ in $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$. For a prime ℓ, write $r := \mathrm{ord}_\ell(|c|)$. Then we have*

(a) *Write $\mathrm{ord}_\ell(|G|) = r + s$, and let $j \geq 0$ be an integer such that G does not contain any proper normal subgroup of order divisible by ℓ^{j+1}. If $r \geq 3s+j$, then*
$$|\,\mathrm{Out}^+(\Pi) \cdot \varphi| \equiv 0 \mod \ell^{\lceil \frac{r-3s-j}{2} \rceil}.$$

(b) *Suppose G is non-abelian and simple. If $\ell^{r+1} \geq |G|^{1/3}$ and G is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ for any q, then*
$$|\,\mathrm{Out}^+(\Pi) \cdot \varphi| \equiv 0 \mod \ell^{\lceil \frac{r}{2} \rceil}.$$

Since the congruences give at best $\equiv 0 \mod |c|$, we find that the results above are only interesting when G is non-abelian (and generated by two elements). Note that when G is abelian, the sizes of the $\mathrm{Out}^+(\Pi)$-orbits of $\varphi : \Pi \to G$ correspond to the indices of certain congruence subgroups inside $\mathrm{SL}_2(\mathbb{Z})$, which are well understood. When $G = \mathbb{Z}/n\mathbb{Z}$, the action is transitive and the size of the single orbit is the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(n)]$; when $G = (\mathbb{Z}/n\mathbb{Z})^2$, the action has $\phi(n)$ components, and the size of each orbit is $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(n)] = |\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})|$.

1.2. *Applications*. From now on we let Π be a free group of rank 2, with basis a, b. Let $X_{\mathrm{SL}_2} := \mathrm{Hom}(\Pi, \mathrm{SL}_2) /\!\!/ \mathrm{GL}_2$ denote the character variety of SL_2-representations of Π, as a scheme of finite type over $\mathbb{Z}$. In preparation for describing the application of the main congruence to Markoff triples, we first recall some relevant properties of X_{SL_2}. See Section 5.2 for more details.

By results of Brumfiel and Hilden, there is an isomorphism $X_{\mathrm{SL}_2} \cong \mathbb{A}^3_{\mathbb{Z}}$, and there is a natural action of $\mathrm{Aut}(\Pi)$ on X_{SL_2} whose induced action on $\mathbb{A}^3_{\mathbb{Z}}$ preserves the hypersurfaces $X_{\mathrm{SL}_2,t}$ given by $x^2 + y^2 + z^2 - xyz - 2 = t$ for any $t \in \mathbb{Z}$. For a ring R and $t \in R$, let $\mathrm{Hom}(\Pi, \mathrm{SL}_2(R))_t$ denote the set of homomorphisms $\varphi : \Pi \to \mathrm{SL}_2(R)$ satisfying $\mathrm{tr}\,\varphi([a,b]) = t$. Such homomorphisms are said to have *trace invariant* t. Then at the level of $\mathbb{F}_q$-points, for any $t \in \mathbb{F}_q - \{2\}$, the isomorphism $X_{\mathrm{SL}_2} \cong \mathbb{A}^3_{\mathbb{Z}}$ induces a *bijection*

$$\begin{aligned} \mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))_t / \mathrm{GL}_2(\mathbb{F}_q) &\longrightarrow X_{\mathrm{SL}_2,t}(\mathbb{F}_q), \\ \varphi &\mapsto (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab)), \end{aligned} \tag{1.1}$$

where $\mathrm{Aut}(\Pi)$ acts naturally on the source and $\mathrm{GL}_2(\mathbb{F}_q)$ acts by conjugation on $\mathrm{SL}_2(\mathbb{F}_q)$.

Let $\Gamma \subset \mathrm{Aut}(\mathbb{A}^3_{\mathbb{Z}})$ be the image of the $\mathrm{Aut}(\Pi)$-action. Then Γ is generated by permutations of the coordinates and the "Vieta" involution $(x, y, z) \mapsto (x, y, xy - z)$. Via the Galois correspondence, the bijection (1.1) provides a dictionary between the geometric properties of the moduli stacks $\mathcal{M}(G)$ for subgroups $G \le \mathrm{SL}_2(\mathbb{F}_q)$ and the properties of the Γ-action on the $\mathbb{F}_q$-points of the surface $X_{\mathrm{SL}_2,t}$.

1.2.1. *Application to Markoff triples*. When $t = -2$, the surface appearing in (1.1) is the *Markoff surface* $\mathbb{X}$ given by[3]

$$\mathbb{X} : x^2 + y^2 + z^2 - xyz = 0,$$

[3] Classically, the Markoff surface is given by $\mathbb{M} : x^2 + y^2 + z^2 - 3xyz = 0$. However, since the map $\mathbb{M} \to \mathbb{X}$ given by $(x, y, z) \mapsto (3x, 3y, 3z)$ is an isomorphism away from 3 and gives a bijection on $\mathbb{Z}$-points, for our purposes it is harmless (and more convenient) to work with $\mathbb{X}$. In Section 5.5, we will make this distinction precise.

which we view as an affine hypersurface in $\mathbb{A}^3_{\mathbb{Z}}$. For primes p, let $\mathbb{X}^*(p) := \mathbb{X}(\mathbb{F}_p) - \{(0,0,0)\}$. It follows from the classification of subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ that for odd p, the bijection (1.1) restricts to give a bijection

$$\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/\,\mathrm{GL}_2(\mathbb{F}_p) \xrightarrow{\sim} \mathbb{X}^*(p). \tag{1.2}$$

This surface first appeared in the work of Markoff [Mar79], [Mar80], where he noted its connection to binary quadratic forms and Diophantine approximation,[4] and proved that

THEOREM 1.2.1 (Markoff [Mar79], [Mar80]). *The positive integer solutions to $\mathbb{X}$ form a single orbit under the action of Γ.*

In [BGS16b], [BGS16a], Bourgain, Gamburd and Sarnak studied the action of Γ on $\mathbb{X}^*(p)$ (for p prime) and conjectured that an analogue of Markoff's result should also hold for its $\mathbb{F}_p$-points:

CONJECTURE 1.2.2 (Bourgain, Gamburd, Sarnak [BGS16b]). *For all primes p, Γ acts transitively on $\mathbb{X}^*(p)$.*

In other words, they conjecture that $\mathbb{X}(\mathbb{F}_p)$ is the union of at most two Γ-orbits: the singleton orbit $\{(0,0,0)\}$, and $\mathbb{X}^*(p)$. (The latter is empty if and only if $p = 3$.) Since $\mathbb{X}(\mathbb{Z})$ contains the solutions $\{(0,0,0),(3,3,3)\}$ and $\mathbb{X}(\mathbb{F}_3) = \{(0,0,0)\}$, a positive solution to this conjecture would imply that for any prime p, the reduction mod p map $\mathbb{X}(\mathbb{Z}) \to \mathbb{X}(\mathbb{F}_p)$ is surjective, in which case, following [BGS16b], we say that $\mathbb{X}$ satisfies *strong approximation* at all primes p.

To the author's knowledge this conjecture was first posed by Baragar in his 1991 PhD thesis [Bar, §V.3]. The first major progress on Conjecture 1.2.2 came in 2016, when Bourgain, Gamburd, and Sarnak [BGS16a] proved

THEOREM 1.2.3 (Bourgain, Gamburd, Sarnak). *The following are true*:

(a) *Let $\mathbb{E}_{\mathrm{bgs}}$ denote the set of primes p for which Γ does not act transitively on $\mathbb{X}^*(p)$. For any $\epsilon > 0$, we have*

$$|\{p \in \mathbb{E}_{\mathrm{bgs}} \mid p \le x\}| = O(x^\epsilon).$$

(b) *Let $\mathcal{C}(p)$ be the largest orbit of Γ on $\mathbb{X}^*(p)$. Then for any $\epsilon > 0$, we have*

$$|\mathbb{X}^*(p) - \mathcal{C}(p)| \le p^\epsilon \qquad \textit{for large } p.$$

Thus, part (a) says that Conjecture 1.2.2 holds for all but a sparse (but possibly infinite) set of primes, and part (b) says that even if it were to fail, it cannot fail too horribly. By the above discussion, the bijection (1.2) yields the following equivalent group-theoretic translation of their result:

[4] See [Bom07] for an excellent exposition of this relationship.

THEOREM 1.2.4 (Bourgain, Gamburd, Sarnak). *For* $t \in \mathbb{F}_p$, *we denote by* $\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_t$ *the subset of surjections with trace invariant* t (*i.e.*, *which satisfy* $\mathrm{tr}\,\varphi([a,b]) = t$).

(a) *Let* $\mathbb{E}_{\mathrm{bgs}}$ *denote the set of primes for which* $\mathrm{Out}(\Pi)$ *does not act transitively on* $\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/\,\mathrm{GL}_2(\mathbb{F}_p)$. *For any* $\epsilon > 0$, *we have*

$$|\{p \in \mathbb{E}_{\mathrm{bgs}} \mid p \le x\}| = O(x^\epsilon).$$

(b) *Let* $\mathcal{C}(p)$ *be the largest orbit of* $\mathrm{Out}(\Pi)$ *on* $\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/\,\mathrm{GL}_2(\mathbb{F}_p)$. *Then for any* $\epsilon > 0$, *we have*

$$|\,\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/\,\mathrm{GL}_2(\mathbb{F}_p) - \mathcal{C}(p)| \le p^\epsilon \qquad \textit{for large } p.$$

By Proposition 5.1.3, for any surjection $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_p)$ with trace invariant -2, $\varphi([a,b])$ must be conjugate to a matrix of the form $\left[\begin{smallmatrix} -1 & u \\ 0 & -1 \end{smallmatrix}\right]$ for $u \in \mathbb{F}_p^\times$. In particular, if $p \ge 3$, then $\varphi([a,b])$ has order $2p$. Then our Corollary 1.1.2(a) implies

THEOREM 1.2.5 (See Theorem 5.5.4). *For every* $p \ge 3$, *every* $\mathrm{Out}^+(\Pi)$-*orbit on* $\mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/\,\mathrm{GL}_2(\mathbb{F}_p)$ *has cardinality divisible by* p. *In particular*, *every* Γ-*orbit on* $\mathbb{X}^*(p)$ *has cardinality divisible by* p.

In particular, every Γ-orbit must have size at least p, so combined with Theorem 1.2.3(b), this establishes Conjecture 1.2.2 for all but a finite set of primes:

THEOREM 1.2.6 (See Theorems 5.5.5, 5.5.7). *The following are true.*

(a) *The* "*exceptional*" *set* $\mathbb{E}_{\mathrm{bgs}}$ *is finite and explicitly bounded.*
(b) *For all primes* $p \notin \mathbb{E}_{\mathrm{bgs}}$, $\mathbb{X}$ *satisfies strong approximation at* p.

Recently, using the above results, Eddy, Fuchs, Litman, Martin, and Tripeny gave an explicit upper bound of $3.448 \cdot 10^{392}$ for the primes in $\mathbb{E}_{\mathrm{bgs}}$ [EFL+23]. In particular, we have effectively reduced the conjecture to a finite computation. The conjecture has been verified for all primes $p < 3000$ by de-Courcy-Ireland and Lee in [dCIL22], so it is reasonable to expect that the final computation will indeed verify Conjecture 1.2.2. If so, then it would follow from Proposition 5.3.3 below that there are no congruence constraints on Markoff numbers mod p other than the ones first noted by Frobenius in 1913 [Fro13], namely that if $p \equiv 3 \mod 4$ and $p \ne 3$, then a Markoff number cannot be $\equiv 0, \frac{\pm 2}{3} \mod p$. Using the work of Meiri, Puder et al. [MP18], one can also deduce strong approximation mod n for most squarefree integers n (Theorem 5.5.8).

Our methods also give congruences for the generalized Markoff equations $x^2 + y^2 + z^2 - xyz = t + 2$.

THEOREM 1.2.7 (See Theorems 5.4.4 and 5.5.9). *Let* $q \ge 3$ *be a prime power. Let* $t \in \mathbb{F}_q - \{2, -2\}$. *Then we may write* $t = \omega + \omega^{-1}$ *for some*

$\omega \in \mathbb{F}_{q^2}^\times - \{-1,1\}$, *and the set* $\{\omega, \omega^{-1}\}$ *is uniquely determined by* t. *Let* $P = (A, B, C)$ *be an* $\mathbb{F}_q$*-point of the hypersurface defined by*

$$x^2 + y^2 + z^2 - xyz = t + 2. \tag{1.3}$$

Let Γ *act on the hypersurface* (1.3) *via the same formulas as for its action on* $\mathbb{X}$. *Suppose at least two of* $\{A, B, C\}$ *are non-zero. Let* ℓ *be an odd prime. Let* $r := \mathrm{ord}_\ell(|\omega|)$, *and write* $\mathrm{ord}_\ell(|\operatorname{SL}_2(\mathbb{F}_q)|) = \mathrm{ord}_\ell(q^3 - q) = r + s$. *Then the* Γ*-orbit of* P *has cardinality* $\equiv 0 \mod \ell^{\max\{r-s,0\}}$.

See Theorem 5.4.4 for a stronger and more precise result. Here, the case $t = -2$ gives the Markoff equation, which was addressed in Theorem 1.2.5, and the case $t = 2$ is called the *Cayley cubic*, which is addressed in [dCIL22, §5]. As announced in [BGS16b], the methods of [BGS16a] also carry over to yield results analogous to Theorem 1.2.3 for these more general equations (1.3) (with an appropriately defined $\mathbb{X}^*(p)$). Thus, combined with Theorem 1.2.7, it is conceivable that conjectures analogous to Conjecture 1.2.2 for these more general equations may also be within reach.

1.2.2. *Application to Nielsen equivalence of generating pairs of finite groups*. In combinatorial group theory the $\mathrm{Out}(\Pi)$-orbits on $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$ are called *Nielsen equivalence classes*. In general, if F_r denotes a free group of rank r, then the problem of Nielsen equivalence asks for a description of the orbits of $\mathrm{Out}(F_r)$ on $\mathrm{Epi}^{\mathrm{ext}}(F_r, G)$. Similarly, one can ask for a description of the orbits of $\mathrm{Out}(F_r)$ on $\mathrm{Epi}(F_r, G)/\operatorname{Aut}(G)$, in which case the orbits are called T_r*-systems* (or sometimes T*-systems*). Such problems arose in the 1950s in the study of group presentations, but have recently garnered renewed interest due to their relevance to the *product replacement algorithm* for generating random elements of finite groups [Pak01], [Lub11]. Let $d(G)$ denote the minimum cardinality of a generating set of G. When $r \geq d(G) + 1$, the expectation is

Conjecture 1.2.8 (See [Gar08, Conj. 1]). *Let* G *be a finite group. If* $r \geq d(G) + 1$, *then* $\mathrm{Out}(F_r)$ *acts transitively on* $\mathrm{Epi}^{\mathrm{ext}}(F_r, G)$.

When G is simple, this conjecture is attributed to Wiegold, and it dates back to the 1970s. The conjecture is known if G is solvable [Dun70], or if $r \geq \log_2(|G|)$ [Lub11, Cor. 3.3]. This latter result can be viewed as an analog of branch/genus stabilization. However, when $r = d(G) = 2$, the action can have arbitrarily many orbits [GS09, Th. 1.8]. The first conjectural complete description of T_2-systems for a family of non-solvable groups was given in 2013 by McCullough and Wanderley [MW13]. For $G = \mathrm{SL}_2(\mathbb{F}_q)$, they conjecture that the trace invariant map

$$\begin{array}{rccc} \tau : & \mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_q)) & \longrightarrow & \mathbb{F}_q \\ & \varphi & \mapsto & \operatorname{tr}\varphi([a,b]) \end{array}$$

completely describes T_2-systems for $\mathrm{SL}_2(\mathbb{F}_q)$:

Conjecture 1.2.9 ("T-classification conjecture" [MW13]). *For any prime power q, the trace invariant map τ induces a bijection from the set of T_2-systems on $\mathrm{SL}_2(\mathbb{F}_q)$ onto its image in $\mathbb{F}_q$.*

Moreover, they give a complete description of the image of τ and computationally verify their conjecture for all $q \leq 131$.

Our Theorem 1.2.6 (also see Theorem 1.2.4) implies that for all but finitely many primes p, there is only one T_2-system on $\mathrm{SL}_2(\mathbb{F}_p)$ with trace invariant $\tau = -2$. More generally, our Corollary 1.1.2 can be viewed as giving non-trivial lower bounds on the sizes of Nielsen equivalence classes of $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$. The full conjecture can be interpreted as an appropriate transitivity property of the Γ-action on the generalized Markoff surfaces (1.3).

1.2.3. *Application to Hurwitz stacks.* In this section we consider the substack $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$ consisting of admissible covers whose Higman invariant has trace -2. For a fixed p, there are two such Higman invariants; they are represented by $\left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & a \\ 0 & -1 \end{smallmatrix}\right]$ where $a \in \mathbb{F}_p^\times$ is a non-square, and they are swapped by the conjugation action of $\mathrm{GL}_2(\mathbb{F}_p)$. Let us write $\mathfrak{c}_1, \mathfrak{c}_2$ for these two conjugacy classes.

From the combinatorial characterization of the components of $\mathcal{A}dm(G)$ described in Section 1.1, we find that for $p \geq 5$, $p \notin \mathbb{E}_{\mathrm{bgs}}$, $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$ is the disjoint union of the two isomorphic components $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{\mathfrak{c}_i}$ for $i = 1, 2$. In particular, each stack $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{\mathfrak{c}_i}$ is *connected*. In accordance with Conjecture 1.2.9, we expect this to hold for *any* conjugacy class of $\mathrm{SL}_2(\mathbb{F}_p)$.

By the coordinatization of the monodromy action given by the character variety, for $i = 1, 2$, we are able to compute the ramification behavior of the coarse scheme of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{\mathfrak{c}_i}$ over $\overline{M(1)}$. Applying Riemann-Hurwitz, this yields an explicit formula for the genus (Theorem 5.6.3). For example, we will show

Theorem 1.2.10 (Theorems 5.6.3 and 5.6.4). *Let p be an odd prime not in the finite set $\mathbb{E}_{\mathrm{bgs}}$. The stacks $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{\mathfrak{c}_i}$ for $i = 1, 2$ are connected and isomorphic to each other. Let $\overline{M_p}$ denote the coarse scheme of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_p))_{\mathfrak{c}_i}$. Then we have*

$$\mathrm{genus}(\overline{M_p}) \sim \frac{1}{12} p^2.$$

An explicit formula for $\mathrm{genus}(\overline{M_p})$ is given in Theorem 5.6.3. In particular, for $p \geq 13$, $\mathrm{genus}(\overline{M_p}) \geq 2$, and for $p = 5, 7, 11$, $\overline{M_p}$ has genus $0, 0, 1$ respectively. It follows by Falting's theorem that for any $p \geq 13$, $p \notin \mathbb{E}_{\mathrm{bgs}}$, and any number field K, only finitely many elliptic curves over K admit an $\mathrm{SL}_2(\mathbb{F}_p)$-cover with ramification index $2p$ defined over K.

We note that the components of $Adm(\mathrm{SL}_2(\mathbb{F}_p))$ can all be obtained by compactifying the quotients of the Poincaré upper half plane by finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ [Che18]; in other words, they are (possibly non-congruence) modular curves. In fact we will show that for all but a density 0 set of primes p, the curves $\overline{M_p}$ are non-congruence (Corollary 5.6.8). From this perspective, the theorem implies

Corollary 1.2.11. *There exist only finitely many non-congruence modular curves of a given genus classifying elliptic curves with* $\mathrm{SL}_2(\mathbb{F}_p)$*-covers, only ramified above the origin with ramification index* $2p$.

This can be viewed as an analog of Rademacher's conjecture, proved by Dennin [DJ75], that there are only finitely many congruence modular curves of a given genus. The same statement is false for general non-congruence modular curves, so to obtain finiteness one needs to restrict the types of non-congruence modular curves considered — for example, by restricting the moduli interpretations. Since congruence modular curves can all be obtained from quotients of components of $Adm(G)$ for abelian groups G, our restriction to components of $Adm(\mathrm{SL}_2(\mathbb{F}_p))$ is not unnatural. Moreover, we expect that the statement of the corollary would continue to hold even without the condition on the ramification index.

1.3. *Organization of the paper.*

1.3.1. *Admissible covers and G-structures.* In Section 2, we give an overview of the moduli stacks we will work with. In Section 2.1, we define admissible G-covers and the stacks $\mathcal{A}dm(G)$, following [ACV03]. In Section 2.2 and Section 2.3, we define the reduced ramification divisor and the Higman invariant of an admissible G-cover and prove some basic results. In Section 2.4, we show that an admissible G-cover can be equivalently characterized as a curve equipped with a G-action satisfying certain properties; this section is not necessary for understanding the rest of the paper. In Section 2.5, we show that there is a homeomorphism $\mathcal{A}dm^0(G) \to \mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli stack of elliptic curves with G-structures [Che18], [Ols12], [PdJ95], [DM69]. Since the forgetful map $\mathcal{M}(G) \to \mathcal{M}(1)$ is finite étale, this leads to a combinatorial characterization of the components of $\mathcal{A}dm(G)$ via Galois theory. This also leads to a natural compactification $\overline{\mathcal{M}(G)}$ of $\mathcal{M}(G)$.

1.3.2. *The main congruence and sketch of the argument.* The purpose of Section 3 is to prove the base form of the main congruence (Theorem 3.5.1), which is the main ingredient in the proof of Theorem 1.1.1. In Sections 3.1–3.4, we review some standard results and describe their extensions to the setting of stacks. In Section 3.5, we prove the main congruence. This congruence essentially comes from noting that for any component $X \subset Adm(G)$, modulo some technical considerations, the pullback of a certain line bundle on $\overline{M(1)}$ to X is an e-th tensor power, where e is the ramification index of the covers classified by X, and hence the forgetful map $X \to \overline{M(1)}$ must have degree which is divisible by e. Here we briefly sketch the argument. We work over an algebraically closed field k of characteristic 0.

We begin by presenting the prototype of the argument. Suppose $f : X \to X(1)$ is a finite map of (connected) smooth proper curves and $E(1) \to X(1)$ is a family of 1-generalized elliptic curves (stable pointed curves of genus 1) with zero section σ'_O. Let $E \to X$ be its pullback via f with section σ_O. Suppose we are given a diagram

$$\begin{tikzcd}
C \arrow[r, "\pi"] \arrow[dr] & E \arrow[r, "\tilde{f}"] \arrow[d, shift right] & E(1) \arrow[d, shift right] \\
& X \arrow[r, "f"] \arrow[u, shift right, bend right, "\sigma_O"'] \arrow[ul, bend left, "\sigma"] & X(1), \arrow[u, shift right, bend right, "\sigma'_O"']
\end{tikzcd}$$

where σ is a section of $C \to X$ contained in the smooth locus of C/X, $\pi \circ \sigma = \sigma_O$, and π is an admissible G-cover, only branched above σ_O, where it has ramification index e. This implies that C/X together with the ramification divisor of π is a stable marked curve, and σ is a component of the ramification divisor. It follows from the étale local description of admissible G-covers (see Section 3.3) that

$$\sigma_O^*\Omega_{E/X} = \sigma^*\pi^*\Omega_{E/X} \cong (\sigma^*\Omega_{C/X})^{\otimes e}. \tag{1.4}$$

Since cotangent sheaves commute with base change, we also have

$$\sigma_O^*\Omega_{E/X} = \sigma_O^*\tilde{f}^*\Omega_{E(1)/X(1)} = f^*\sigma_O'^*\Omega_{E(1)/X(1)}.$$

Let $\lambda := \sigma_O'^*\Omega_{E(1)/X(1)}$. Then since f is finite flat, taking degrees gives

$$\deg f \cdot \deg \lambda = \deg f^*\lambda = e \cdot \deg(\sigma^*\Omega_{C/X}).$$

Since degrees are integers, this yields

$$\deg f \equiv 0 \mod \frac{e}{\gcd(e, \deg \lambda)}. \tag{1.5}$$

This is the essence of the main congruence. We wish to run this prototype argument when $E(1) \to X(1)$ is the universal family $\overline{\mathcal{E}(1)} \to \overline{\mathcal{M}(1)}$, X is a component $\mathcal{X} \subset \mathcal{A}dm(G)$, f is the forgetful map $\mathfrak{f} : \mathcal{X} \to \overline{\mathcal{M}(1)}$, and $\pi : C \to E$ is the universal family $\pi : \mathcal{C} \to \mathcal{E}$ over $\mathcal{X}$. In this case, we find that λ is the Hodge bundle, which has degree $\frac{1}{24}$ (Proposition 3.4.9). However, in this setting we are immediately presented with two difficulties:

(a) The universal admissible G-cover $\pi : \mathcal{C} \to \mathcal{E}$ may not admit a ramified section σ.

The reduced ramification divisor $\mathcal{R}_\pi$ of the universal admissible G-cover $\pi : \mathcal{C} \to \mathcal{E}$ is always finite étale over $\mathcal{X}$, but it may not admit a section. If $\mathcal{R} \subset \mathcal{R}_\pi$ is a component, the base change of $\pi : \mathcal{C} \to \mathcal{E}$ to $\mathcal{R}$ admits a ramified section σ, and $\mathcal{R}$ is the minimal extension of $\mathcal{X}$ for which this happens. Thus we can try to apply the prototype argument to the base change $\pi : \mathcal{C}_{\mathcal{R}} \to \mathcal{E}_{\mathcal{R}}$, at the cost of potentially weakening the congruence. Let $d_{\mathcal{X}}$ denote the degree of the map on coarse schemes $R \to X$ induced by $\mathcal{R} \to \mathcal{X}$.

(b) Since $\mathcal{R}$ is a stack, the degree of $\sigma^*\Omega_{\mathcal{C}_{\mathcal{R}}/\mathcal{R}}$ may not be an integer.

Since $\mathcal{X}$ is Deligne-Mumford, at least we have $\deg \sigma^*\Omega_{\mathcal{C}_{\mathcal{R}}/\mathcal{R}} \in \mathbb{Q}$. For a geometric point $x : \operatorname{Spec} k \to \mathcal{X}$, $\sigma^*\Omega_{\mathcal{C}_{\mathcal{R}}/\mathcal{R}}$ defines a rank 1 representation of $\operatorname{Aut}_{\mathcal{X}}(x)$, which we call the *local character* at x. By a theorem of Olsson [Ols12], the denominator of the rational number $\deg \sigma^*\Omega_{\mathcal{C}_{\mathcal{R}}/\mathcal{R}}$ can be bounded in terms of the orders of the local characters. Since $\mathcal{X}$ is Noetherian, let $m_{\mathcal{X}}$ denote the minimum positive integer such that $(\sigma^*\Omega_{\mathcal{C}_{\mathcal{R}}/\mathcal{R}})^{\otimes m_{\mathcal{X}}}$ has trivial local characters.

The integers $d_{\mathcal{X}}, m_{\mathcal{X}}$ defined above quantify the obstructions to achieving a congruence of the form (1.5). As described in Section 4.11, both obstructions can be non-trivial and can affect the congruence. Since the forgetful map $\mathfrak{f} : \mathcal{X} \to \overline{\mathcal{M}(1)}$ is generally not representable, hence not finite, we will phrase the congruence in terms of the induced finite flat map on coarse schemes $f : X \to \overline{M(1)}$. The purest form of our main result is

THEOREM 1.3.1 (Main congruence; see Theorem 3.5.1). *Let $\mathcal{X} \subset \mathcal{A}dm(G)$ be a connected component classifying G-covers of elliptic curves with ramification index e above the origin. Let $d_{\mathcal{X}}, m_{\mathcal{X}}$ be as above. Then the forgetful map $f : X \to \overline{M(1)}$ satisfies*

$$\deg(X \xrightarrow{f} \overline{M(1)}) \equiv 0 \mod \frac{12e}{\gcd(12e, m_{\mathcal{X}} d_{\mathcal{X}})}$$

In this form, the congruence is difficult to use, since one must first check that $m_{\mathcal{X}}, d_{\mathcal{X}}$ do not share too many divisors with e. Thus we are motivated to give more easily accessible bounds for $m_{\mathcal{X}}, d_{\mathcal{X}}$. If $c \in G$ represents the Higman invariant of covers classified by $\mathcal{X}$, then by Proposition 3.2.2, $d_{\mathcal{X}}$ must divide $|C_G(c)/\langle c\rangle|$.[5] The integer $m_{\mathcal{X}}$ is related to the order of automorphism groups of geometric points of $\mathcal{R}$. For points corresponding to smooth covers, these automorphism groups are contained in the center $Z(G)$ (see Proposition 3.5.2). For points lying over the boundary of $\mathcal{A}dm(G)$, the possible non-irreducibility of

[5] This bound is probably not optimal in general, but it will be enough for our main application to the Markoff equation.

degenerate covers complicates the situation; this issue is addressed in Section 4, which we describe next.

1.3.3. *Combinatorial characterization of the cusps*. Borrowing terminology from the classical theory of the moduli of elliptic curves, we call points lying on the boundary *cusps*, and we call the objects they correspond to *cuspidal objects*. In Section 4 we give a combinatorial characterization of cuspidal objects using Galois theory. In Section 4.8, we will attach to any cuspidal object a "δ-invariant," which is the equivalence class of an element of $G \times G$. In Section 4.10, we calculate the automorphism group of a cuspidal object from its δ-invariant, which allows us to control, in purely combinatorial terms, the integer $m_{\mathcal{X}}$ described above. Together with the bound on $d_{\mathcal{X}}$ mentioned above, we give a purely combinatorial corollary of the main congruence (Theorem 4.10.5). In Section 4.12 we deduce the congruences of Theorems 1.1.1 and 1.1.2 from this combinatorial statement.

1.3.4. *Applications to Markoff triples and the geometry of* $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))$. In Section 5 we prove the applications of our congruences described in Section 1.2. In Section 5.2 we describe the theory of the character variety for SL_2-representations of a free group of rank 2, and we make precise the connection between the stacks $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))$ and the Markoff equation. In Section 5.3, we use the connection with the character variety to show that the automorphism groups of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))$ are as small as possible: they are reduced to the center $Z(\mathrm{SL}_2(\mathbb{F}_q))$. In Section 5.4 we use this calculation as input to the main congruence and obtain congruences for the degrees of components of $Adm(\mathrm{SL}_2(\mathbb{F}_q))$ over $\overline{M(1)}$, or equivalently the sizes of Γ-orbits of $\mathbb{F}_q$-points on the associated generalized Markoff surface. In Section 5.5 we bring everything together and prove Theorem 1.2.6. In Section 5.6, we give formulas for the genus of the components of $Adm(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$ for $p \notin \mathbb{E}_{\mathrm{bgs}}$, and show that they are non-congruence for a density 1 set of primes.

1.4. *Related work*.

1.4.1. *Integral points on more general character varieties*. A related problem is to understand the mapping class group orbits on the *integral* points of character varieties. In the case of SL_2-representations of the fundamental group of one-holed torus with trace invariant -2, this question is resolved by a classical result of Markoff [Mar79] (see Theorem 1.2.1). In particular, in this case we obtain five orbits, represented by $(0,0,0)$, $(3,3,3)$, $(3,-3,-3)$, $(-3,3,-3)$, and $(-3,-3,3)$. This should be viewed as a type of "finite generation" result on the set of integral points up to the action of the mapping class group. An analogous finite generation result for integral points for more general character varieties is proven in [Wha20] using techniques from differential geometry.

1.4.2. *Teichmüller curves*. For a component $M \subset M(G)_{\mathbb{C}}$, let g denote the genus of the covers it classifies. Forgetting the base curve gives a natural map $M \to M_g$, whose image is a *Teichmüller curve* as first studied by Veech [Vee89] in the context of dynamics of billiard tables (also see [Che17], [HS06], [Loc05], [MT02], [Zor06]). Specifically, these are Teichmüller curves generated by a square-tiled surface (called "origami curves" in [HS09], [Sch04]). If M is a component of $M(G)_{\mathbb{C}}/\operatorname{Out}(G)$, then it follows from [Sch04] that M is the quotient of the upper half plane by the *Veech group* of the corresponding square tiled surface. In this language, our congruence can be interpreted as a congruence on the index of the Veech group inside $\mathrm{SL}_2(\mathbb{Z})$. For genus $g = 2$, the Teichmüller curves in M_2 have been studied extensively in [McM03], [McM05], [Dur18]. In [McM05] McMullen faced a similar issue of connectedness of a certain moduli space, which he was able to solve by reducing it to a problem in combinatorial number theory. It can be shown that his moduli space is a subspace of $M(S_d)/\operatorname{Out}(S_d) \sqcup M(A_d)/\operatorname{Out}(A_d)$, where A_d (resp. S_d) denotes the alternating (resp. symmetric group) on d letters. Specifically, [McM05, Cor. 1.5] can be viewed as saying that for $d \geq 4$, the subscheme of $M(S_d)/\operatorname{Out}(S_d) \sqcup M(A_d)/\operatorname{Out}(A_d)$ consisting of covers with Higman invariant the class of a 3-cycle has one or two components, according to whether d is even or odd.

1.4.3. *Tamely ramified covers of $\mathbb{P}^1$ in characteristic p*. In [BBCL22], for every prime ℓ, we realized infinitely many alternating and symmetric groups as quotients of the tame fundamental group of $\pi_1(\mathbb{P}^1_{\mathbb{F}_\ell} - \{0, 1, \infty\})$. This work is in the spirit of a tame version of "Abhyankar's conjecture" [HOPS18]. The precise result stated in [BBCL22, Th. 3.5.1] only holds for those primes p for which Γ acts transitively on $\mathbb{X}^*(p)$. Thus, Theorem 1.2.6 of the present paper can be viewed as a strengthening of [BBCL22, Th. 3.5.1].

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1.6. *Notation and conventions*. In our usage of stacks, we will follow the definitions of the stacks project [Stacks, 026O]. We will typically use script letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ to denote stacks and use Roman letters $X, Y, Z, \ldots$ to denote schemes. Typically, X will be the coarse scheme of $\mathcal{X}$.

The symbol π_1 applied to a geometric object will by default refer to the étale fundamental group. If X is a topological space and $x \in X$, then $\pi_1^{\mathrm{top}}(X, x)$

denotes its topological fundamental group. We take the convention that if γ_1, γ_2 are two loops in X based at x, then the product $\gamma_1\gamma_2$ in $\pi_1^{\mathrm{top}}(X, x)$ is represented by the loop that first follows γ_2 and then follows γ_1. This is consistent with our use of Galois theory, where we adopt the convention that Galois actions are right actions and monodromy actions are left actions. In particular, the étale fundamental group is the automorphism group of a fiber functor and hence acts on fibers from the left.

For elements a, b of a group G, $a \sim b$ means that a is conjugate to b, $[a, b] := aba^{-1}b^{-1}$ denotes the commutator, $a^b := b^{-1}ab$, and ${}^b a := bab^{-1}$. The order of $a \in G$ is denoted $|a|$.

Here is a summary of our notation:

- G will always be a finite group; $\mathrm{Cl}(G)$ denotes the set of conjugacy classes of G.
- I will generally denote the matrix $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Thus the center of $\mathrm{SL}_2(\mathbb{F}_q)$ is $\{\pm I\}$.
- Given groups A and B, $\mathrm{Epi}^{\mathrm{ext}}(A, B) := \mathrm{Epi}(A, B)/\operatorname{Inn}(B)$ is the set of conjugacy classes of surjections $A \to B$ (called "exterior epimorphisms" in [DM69]).
- $\mathbb{S}$ is the universal base scheme. Often we will take $\mathbb{S} = \operatorname{Spec} \mathbb{Z}[1/|G|]$.
- $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$.
- $\mathcal{M}(1)$ is the moduli stack of elliptic curves, and $\overline{\mathcal{M}(1)}$ is the compactification of $\mathcal{M}(1)$ by stable curves.
- $\mathcal{M}(G)$ is the moduli stack of elliptic curves with G-structures. This is finite étale over $\mathcal{M}(1)$, but does not always carry a universal family of covers.
- $\mathcal{A}dm(G)$ is the moduli stack of admissible G-covers of 1-generalized elliptic curves (i.e., stable pointed curves of genus 1). This carries a universal family of covers, but is typically not finite over $\overline{\mathcal{M}(1)}$.
- $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ is the open substack classifying smooth covers. It is an étale gerbe over $\mathcal{M}(G)$ with group $Z(G)$. In particular, $\mathcal{A}dm^0(G)$ and $\mathcal{M}(G)$ have the same coarse schemes.
- $\overline{\mathcal{M}(G)}$ denotes the *rigidification* $\mathcal{A}dm(G) /\!\!/ Z(G)$ (Definition 2.5.9).

2. Admissible G-covers of elliptic curves

In this section we recall the theory of admissible G-covers and their moduli stacks $\mathcal{A}dm(G)$, following [AV02], [ACV03], [Ols07], and we will relate them to the moduli stacks $\mathcal{M}(G)$ of elliptic curves with G-structures as studied in [Che18] (also see [DM69, §5] and [PdJ95]). Here G will always be a finite group. We will work universally over the base scheme $\mathbb{S} := \operatorname{Spec} \mathbb{Z}[1/|G|]$.

2.1. *Admissible G-covers and their moduli.*

Definition 2.1.1. A morphism of schemes $f : C \to S$ is a *nodal curve* if f is flat, proper, of finite presentation, and whose geometric fibers are of pure dimension 1 and whose only singularities are ordinary double points. The morphism $f : C \to S$ is a *prestable curve* if f is a nodal curve with connected geometric fibers.

Definition 2.1.2. A *prestable n-pointed curve of genus g* is a prestable curve $f : C \to S$ with geometric fibers of arithmetic genus g, equipped with n mutually disjoint sections $\sigma_1, \dots, \sigma_n : S \to C$ lying in the smooth locus $C_{\mathrm{sm}} \subset C$. The sections $\sigma_1, \dots, \sigma_n$ are called *markings*.

The prestable n-pointed curve $(C, \{\sigma_i\})$ is a *stable n-pointed curve* if for any geometric point $\overline{s} : \operatorname{Spec} k \to S$, the geometric fiber $C_{\overline{s}}$ satisfies any of the following equivalent conditions [Man99, §5, Lemma 1.2.1]:

- $C_{\overline{s}}$ has only finitely many k-automorphisms which fix the sections $\sigma_1, \dots, \sigma_n$;
- for any irreducible component $Z \subset C_{\overline{s}}$ with normalization Z', if Z' has genus 0, then it must contain at least three special[6] points, and if Z' has genus 1, then it must contain at least one special point;
- $\omega_{C_{\overline{s}}}(\sum_i \sigma_i)$ is ample, where $\omega_{C_{\overline{s}}}$ is the dualizing sheaf (see Section 3.1.2).

A *prestable marked curve* is a prestable curve $f : C \to S$ equipped with an effective Cartier divisor $R \subset C_{\mathrm{sm}}$, finite étale over S, called a *marking*. The pair $(C/S, R)$ is a *stable marked curve* if for every geometric point $\overline{s} : \operatorname{Spec} k \to S$, $C_{\overline{s}}$ has only finitely many k-automorphisms which preserve R.

A point in C is called a node if it is the image of a node of a geometric fiber of C/S. A point in C is special if it is either a node or in a marking. The *generic locus* of a (pre)stable n-pointed curve $f : C \to S$ is the open complement of the special points, denoted $C_{\mathrm{gen}} \subset C$.

Definition 2.1.3. A 1-*generalized elliptic curve* is a stable 1-pointed curve of genus 1. We will denote the section by σ_O and the divisor it defines by O. We will write E° for the open subscheme $E - O$. The moduli stack of 1-generalized elliptic curves is denoted $\overline{\mathcal{M}(1)}$. The open substack classifying (smooth) elliptic curves is denoted $\mathcal{M}(1)$.[7] Let $M(1) \cong \mathbb{A}^1_{\mathbb{S}}$ and $\overline{M(1)} \cong \mathbb{P}^1_{\mathbb{S}}$ denote their coarse schemes (see Definition 2.1.8 below), where the isomorphisms are given by the j-invariant.

[6] For an irreducible component $Z \subset C_{\overline{s}}$ with normalization Z', we say that a point $z \in Z'$ is special if either it maps to a marking or a node in $C_{\overline{s}}$.

[7] Here, the "1" in "$\mathcal{M}(1)$" indicates "no level structures" (or "trivial level structures"). In general $\mathcal{M}(G)$ will denote the moduli stack of elliptic curves with G-structures (cf. Section 2.5 below).

For a scheme X and a geometric point $\overline{p} : \operatorname{Spec} k \to X$, let $\mathcal{O}_{X,\overline{p}}$ be the strict henselization of the local ring (or just the strict local ring) of X at the image p of $\overline{p}$. Its residue field is the separable closure of the residue field $\kappa(p)$ inside k. For the purposes of the following definition, we will use the notation

$$X_{(\overline{p})} := \operatorname{Spec} \mathcal{O}_{X,\overline{p}},$$

and we say that $X_{(\overline{p})}$ is the strict localization of X at p.

Definition 2.1.4 (cf. [ACV03, Def. 4.3.1]). An *admissible G-cover* of a prestable n-pointed curve $(D \xrightarrow{f} S, \{\sigma_i\})$ is a finite map $\pi : C \to D$ equipped with a *right* action of G on C leaving π invariant, where

(1) $C \to S$ is a prestable curve.
(2) Every node of C maps to a node of D.
(3) The restriction of π to the preimage of D_{gen} is a G-torsor.
(4) Let $\overline{p} : \operatorname{Spec} k \to C$ be a geometric point whose image in D lands in a marking. Let $\overline{s} := f(\pi \circ \overline{p})$ be its image in S, with strict local ring $\mathcal{O}_{S,\overline{s}}$. For some integer $e \geq 1$, let

$$\pi' : C' := \operatorname{Spec} \mathcal{O}_{S,\overline{s}}[\xi] \longrightarrow D' := \operatorname{Spec} \mathcal{O}_{S,\overline{s}}[x]$$

be given by $x \mapsto \xi^e$. Let $\overline{p}' : \operatorname{Spec} k \to C'$ be a geometric point with image the point $\xi = 0$. Then for some choice of e as above, there is a commutative diagram

$$\begin{array}{ccccc} C_{(\overline{p})} & \xrightarrow{\pi} & D_{(\pi\circ\overline{p})} & \xrightarrow{f} & S_{(\overline{s})} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| \\ C'_{(\overline{p}')} & \xrightarrow{\pi'} & D'_{(\pi'\circ\overline{p}')} & \xrightarrow{f'} & S_{(\overline{s})}, \end{array}$$

where the vertical maps are isomorphisms with the marking corresponding to $x = 0$, and f' is induced by the structure map $D' \to S_{(\overline{s})}$. We call this integer e the *ramification index* at $\overline{p}$.
(5) Let $\overline{p} : \operatorname{Spec} k \to C$ be a geometric point whose image in D is a node, and let $\overline{s} := f(\pi(\overline{p}))$ be its image in S. For some integer $r \geq 1$ and a in the maximal ideal $\mathfrak{m}_{S,\overline{s}} \subset \mathcal{O}_{S,\overline{s}}$, let

$$\pi' : C' := \operatorname{Spec} \mathcal{O}_{S,\overline{s}}[\xi, \eta]/(\xi\eta - a) \longrightarrow D' := \operatorname{Spec} \mathcal{O}_{S,\overline{s}}[x, y]/(xy - a^r)$$

be given by $(x, y) \mapsto (\xi^r, \eta^r)$. Let $p' : \operatorname{Spec} k \to C'$ be a geometric point with image $(\xi, \eta) = (0, 0)$. Then there is a commutative diagram

$$\begin{array}{ccccc} C_{(\overline{p})} & \xrightarrow{\pi} & D_{(\pi\circ\overline{p})} & \xrightarrow{f} & S_{(\overline{s})} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| \\ C'_{(\overline{p}')} & \xrightarrow{\pi'} & D'_{(\pi'\circ\overline{p}')} & \xrightarrow{f'} & S_{(\overline{s})}, \end{array}$$

where the vertical maps are isomorphisms, and f' is induced by the structure map $D' \to S_{(\overline{s})}$.

(6) If $\overline{p}$ is a geometric point landing in a node of C with image $\overline{s} \in S$, then in the notation of (5) applied to the fiber $C_{\overline{s}}$, the stabilizer $G_{\overline{p}} := \mathrm{Stab}_G(\overline{p})$ is cyclic and the action of a generator $g \in G_{\overline{p}}$ on $(C_{\overline{s}})_{(\overline{p})}$ is given étale locally by sending $\xi \mapsto \zeta\xi, \eta \mapsto \zeta^{-1}\eta$ for some primitive e-th root of unity $\zeta \in k(\overline{s})$. Here we say that the node $\overline{p}$ is *balanced*.

An admissible G-cover $\pi : C \to D$ is *smooth* if C/S is smooth.

Remark 2.1.5. Here we record some observations about admissible G-covers.

(a) We note that part (2) of the definition is implied by (3) and (4). It follows from (5) that admissible G-covers map smooth points to smooth points. Thus, for an admissible G-cover $\pi : C \to D$, a point $x \in C$ is a node if and only if $\pi(x) \in D$ is a node. In particular, C/S is smooth if and only if D/S is smooth.

(b) By part (3), G acts transitively on the geometric fiber above any given marked point $x \in D$. In particular, the ramification indices at points lying over x are all equal. If D has a single marking and S is connected, then all points above the marked section have the same ramification, which we simply call the ramification index of the cover.

(c) Our definition of admissible covers differs from the definition used in Abramovich-Corti-Vistoli [ACV03, Def. 4.3.1] in that we require the covering curve C to be prestable, and hence it has *connected* geometric fibers, whereas [ACV03] only requires that C be a nodal curve, so its geometric fibers can be disconnected. An *ACV-admissible G-cover* is a map $\pi : C \to D$ where C/S is a nodal curve and π satisfies conditions (2)–(6) of the definition. Thus an admissible G-cover is equivalently an ACV-admissible G-cover with connected geometric fibers.

(d) A finite map $\pi : C \to E$ is an *admissible cover* if it satisfies (1), (2), (3), (4), and (5). If it admits a G-action leaving π invariant which moreover satisfies (6), then we say that the G-action is "balanced." Thus, an admissible G-cover is an admissible cover admitting a balanced G-action. In general, for a cover satisfying (2), the action of a generator of G_p on the local ring of a node p can send $\xi \mapsto \zeta\xi$, $\eta \mapsto \zeta'\eta$, where ζ, ζ' are arbitrary primitive r-th roots of unity. However if $\zeta' \neq \zeta^{-1}$, this would force $a = 0$ in the notation of (5), and so the node cannot appear in a generically smooth family. Since we will be interested in compactifications of the moduli stack of *smooth admissible G-covers*, it suffices to restrict our attention to balanced actions.

(e) Let $\overline{p}$ be a geometric point landing in a node of C with image $\overline{s} \in S$. By Proposition 6.1.4, the balanced condition at $\overline{p}$ can be equivalently phrased as follows: The normalization map $\nu : C'_{\overline{s}} \to C_{\overline{s}}$ induces a decomposition of the cotangent space $T^*_{C_{\overline{s}},\overline{p}}$ into a sum of two 1-dimensional subspaces (the branches of the node). The G-action is *balanced* at $\overline{p}$ if the (left) action of $G_{\overline{p}} := \mathrm{Stab}_G(\overline{p})$ on this cotangent space preserves this decomposition and acts faithfully via mutually inverse characters on each summand.

Definition 2.1.6. Given admissible (resp. ACV-admissible) G-covers of genus g stable n-pointed curves $(C \to D \to S, \{\sigma_i\}_{1\le i\le n})$ and $(C' \to D' \to S', \{\sigma'_i\}_{1\le i\le n})$, a morphism $(C' \to D' \to S', \{\sigma_i\}) \to (C \to D \to S, \{\sigma'_i\})$ is a diagram

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ D' \arrow[r, "\overline{f}"] \arrow[d] & D \arrow[d] \\ S' \arrow[r] & S, \end{tikzcd}$$

where all squares are cartesian, $\overline{f}$ sends each σ_i to σ'_i, and f is G-equivariant. Since $C \to D, C \to S$ are epimorphisms [Stacks, 023Q], any such diagram is determined by a morphism $f : C' \to C$. Moreover, we will see (Proposition 2.4.6) that an admissible G-cover $C \to D$ induces an isomorphism $C/G \cong D$, so any G-equivariant map $f : C' \to C$ determines a diagram as above, whence a morphism of admissible (resp. ACV-admissible) G-covers. The category of ACV-admissible G-covers of stable n-pointed curves of genus g is fibered in groupoids over $\underline{\mathbf{Sch}}/\mathbb{S}$. In [ACV03, §4.3], this category is denoted $\mathcal{A}dm_{g,n}(G)$. The full subcategory of admissible G-covers forms an open and closed substack $\mathcal{A}dm^{\mathrm{conn}}_{g,n}(G) \subset \mathcal{A}dm_{g,n}(G)$. In our case, since we will only consider connected covers of 1-generalized elliptic curves, we will abbreviate

$$\mathcal{A}dm(G) := \mathcal{A}dm^{\mathrm{conn}}_{1,1}(G).$$

Let $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ denote the open substack consisting of smooth covers. If $\phi : \mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ denotes the functor sending an admissible cover $C \to E \to S$ to $E \to S$, then we have $\mathcal{A}dm^0(G) = \phi^{-1}(\mathcal{M}(1))$. As usual let $Adm(G), Adm^0(G)$ denote the corresponding coarse spaces (see Theorem 2.1.11 below).

Remark 2.1.7. Beware that despite the nomenclature, given admissible G-covers $\pi : C \to E$, $\pi' : C' \to E$ of the same 1-generalized elliptic curve E, a morphism of admissible G-covers $\pi \to \pi'$ *need not* induce the identity on E. In other words, morphisms of admissible G-covers are not necessarily "morphisms of covers." In this sense, it may be better to think of an admissible G-cover as the curve C equipped with a G-action and a marking divisor. This perspective is taken in Section 2.4 below.

Definition 2.1.8. Recall that the coarse space of an algebraic stack $\mathcal{X}$ is a map $c : \mathcal{X} \to X$ with X an algebraic space which satisfies

(a) Any map $\mathcal{X} \to T$ with T an algebraic space factors uniquely through $c : \mathcal{X} \to X$.
(b) For any algebraically closed field k, c induces a bijection of sets $\mathcal{X}(k)/\cong \xrightarrow{\sim} X(k)$.

The coarse space, if it exists, is uniquely determined by (a). If $\mathcal{X}$ is locally of finite presentation over $\mathbb{S}$ and has finite inertia (e.g., if it is a separated and locally finitely presented Deligne-Mumford stack), then the coarse space always exists and enjoys the following additional properties, which we will use repeatedly without mention.

THEOREM 2.1.9 (Keel-Mori theorem). *Let $\mathcal{X}$ be an algebraic stack locally of finite presentation* (*over* $\mathbb{S}$) *with finite inertia. Then there exists a coarse space $c : \mathcal{X} \to X$ such that moreover we have*

(a) *$c : \mathcal{X} \to X$ is proper*, *quasi-finite*, *and a universal homeomorphism*;
(b) *formation of X commutes with flat base change*;
(c) $c_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$;
(d) *if $\mathcal{X}$ is separated over* $\mathbb{S}$, *then X is also separated over* $\mathbb{S}$;
(e) *if* $\mathbb{S}$ *is locally Noetherian*, *then X is locally of finite presentation over* $\mathbb{S}$;
(f) *if* $\mathbb{S}$ *is locally Noetherian and $\mathcal{X}$ is proper over* $\mathbb{S}$, *then X is also proper over* $\mathbb{S}$.

Proof. See [Stacks, 04XE] for the definition of the topological space of a stack. By [Stacks, 0DUT], c is separated, quasi-compact, and a universal homeomorphism, and it commutes with flat base change. Since $\mathcal{X}$ is locally of finite type, the same is true of c, so it is of finite type. This implies that c is proper and quasi-finite [Stacks, 0G2M]. This establishes (a) and (b). Part (c) follows from (b) and the universal property of coarse spaces (see [AV02, Th. 2.2.1]). Part (d) is [Stacks, 0DUY]. Part (e) is [Stacks, 0DUX]. Part (f) follows from (a), (d), and (e). □

In relative dimension 1, smoothness of $\mathcal{X}$ often implies smoothness of X.

LEMMA 2.1.10. *Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack over a regular Noetherian scheme S whose fibers have pure dimension* 1. *Suppose its coarse space X is a scheme. Then X is smooth and proper over S.*

Proof. By Theorem 2.1.9(f), X is proper over S, so it remains to establish smoothness. By the local structure of Deligne-Mumford stacks [Ols16, Th. 11.3.1], $\mathcal{X}$ admits an étale covering by schemes $\{U_i \to \mathcal{X}\}$ such that $\mathcal{X} \times_X U_i \cong [V_i/G_i]$ for some finite U_i-scheme V_i equipped with an action of a finite group G_i. From the proof we may even take V_i to be affine, and hence $U_i = V_i/G_i$ is also affine. Since $V_i \to \mathcal{X}$ is étale, each V_i is a smooth affine curve over S. By [KM85, Th. on p. 508], the quotients $U_i = V_i/G_i$ are also smooth affine curves over S, so X is smooth over S [Stacks, 036U]. □

THEOREM 2.1.11 ([AV02], [ACV03]). *The following hold*:

(a) *The category $\mathcal{A}dm(G)$ is a smooth proper Deligne-Mumford stack of pure dimension* 1 (*over* $\mathbb{S}$). *In particular, it has finite diagonal.*

(b) *The natural map $\phi : \mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ sending an admissible cover $C \to E \to S$ to $E \to S$ is flat, proper, and quasi-finite.*[8] *In particular, $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ is open and dense.*

(c) *$\mathcal{A}dm(G)$ admits a coarse scheme $Adm(G)$ satisfying the properties of Theorem 2.1.9, and the map $Adm(G) \to \overline{M(1)}$ induced by ϕ is finite. In particular, $Adm(G)$ is a scheme.*

Proof. The statements are preserved by base change, so we may assume $\mathbb{S}$ is Noetherian. In [ACV03], these facts are proven for the stack $\mathcal{B}_{1,1}^{\mathrm{bal}}(G)$ of *twisted G-covers* of 1-generalized elliptic curves, which is equivalent to $\mathcal{A}dm_{1,1}(G)$ [ACV03, Th. 4.3.2], and so the theorem follows from the fact that $\mathcal{A}dm(G)$ is an open and closed substack of $\mathcal{A}dm_{1,1}(G)$.

Specifically, that $\mathcal{A}dm(G)$ is a proper Deligne-Mumford stack follows from [ACV03, Th. 2.1.7](1) (also see [AV02, Th. 1.4.1]), which also implies the finiteness of the diagonal.[9] The smoothness and 1-dimensionality of $\mathcal{A}dm(G)$ follows from [ACV03, Th. 3.0.2]. This proves (a).

The properties of the map ϕ follows from [ACV03, Cor. 3.0.5]. To see that $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ is open dense, note that a flat locally of finite presentation morphism of algebraic stacks induces an open map of topological spaces [Stacks, 06R7], so the map ϕ is open. If $\mathcal{X} \subsetneq \mathcal{A}dm(G)$ is a proper closed substack containing $\mathcal{A}dm^0(G)$, then its complement $\mathcal{U} \subset \mathcal{A}dm(G)$ is open and non-empty, which maps onto an open substack of $\overline{\mathcal{M}(1)}$ containing the cusp, which must intersect $\mathcal{M}(1)$ non-trivially since $\mathcal{M}(1) \subset \overline{\mathcal{M}(1)}$ is open dense. But this contradicts the fact that $\mathcal{A}dm^0(G) = \phi^{-1}(\mathcal{M}(1))$.[10] This proves (b).

For (c), note that finite diagonal implies finite inertia, so the existence and properties of the coarse space follows from Theorem 2.1.9. The rest of part (c) follows from [AV02, Th. 1.4.1] (also see [ACV03, §2.2]), where in their notation they show that the map

[8] Quasi-finiteness for morphisms of algebraic stacks is defined in [Stacks, 0G2L] (also see [Vis89, Def. 1.8]).

[9] A proper morphism is separated, so $\mathcal{A}dm(G)$ has proper diagonal. Since $\mathcal{A}dm(G)$ is Deligne-Mumford, its diagonal is also unramified, hence locally quasi-finite, hence finite by Zariski's main theorem [Stacks, 0A4X].

[10] This denseness also follows from the deformation theory (see Proposition 2.5.3).

$$B_{1,1}(G) = \mathbf{K}_{1,1}(BG, 0) \longrightarrow \mathbf{K}_{1,1}(\mathbb{S}, 0) = \overline{M(1)}$$

is finite, where $B_{1,1}(G)$ is the coarse space of $\mathcal{B}_{1,1}(G)$. Since $\mathcal{B}_{1,1}^{\mathrm{bal}}(G)$ is an open and closed substack of $\mathcal{B}_{1,1}(G)$, $Adm(G)$ is also open and closed inside $B_{1,1}(G)$, so $Adm(G) \to \overline{M(1)}$ is finite as desired. Since $\overline{M(1)}$ is a scheme, the finiteness also implies that $Adm(G)$ is a scheme [Stacks, 03ZQ]. (One could also use [Stacks, 03XX].) □

2.2. *The reduced ramification divisor of an admissible G-cover*. Let $\pi : C \to D$ be an admissible G-cover of a stable n-pointed curve D over S. Here we will define its *reduced ramification divisor* $\mathcal{R}_\pi$, which is a closed subscheme of C, supported on the smooth points of C with non-trivial inertia groups, and is finite étale over S. As an S-scheme it is determined up to isomorphism by this property.[11] Moreover it inherits an action of G, and its connected components can be controlled by group-theoretic properties of G. Our discussion follows [BR11, §4.1.2]. Note that the reduced ramification divisor never contains any nodes, even when they have nontrivial G-stabilizers. We do not consider nodes to be ramification points.

For a non-trivial cyclic subgroup $H \leq G$, let $C^H \subset C$ denote the closed subscheme of fixed points. Namely, for $h \in H$, let $C^h \hookrightarrow C$ denote the equalizer of the maps $\mathrm{id}, h : C \rightrightarrows C$, which is a closed subscheme of C since C is separated.[12] Let $C^H := \bigcap_{h \in H} C^h$ be the scheme theoretic intersection [Stacks, 0C4H]. Note that if $h \in H$ is a generator, then we have $C^H = C^h$. If $U := \operatorname{Spec} A \subset C$ is a G-invariant open affine, then $C^H \cap U = \operatorname{Spec} A_H$, where A_H is the ring of coinvariants $A/\langle \{ha - a\}_{a \in A, h \in H}\rangle$.

Proposition 2.2.1. *Let $\pi : C \to D$ be an admissible G-cover of a stable n-pointed curve D over S. Let $C_{\mathrm{sm}} \subset C$ denote the smooth locus of C/S. For a non-trivial cyclic subgroup $H \leq G$, let $C_{\mathrm{sm}}^H := C^H \cap C_{\mathrm{sm}} = (C_{\mathrm{sm}})^H$. Then either C_{sm}^H is empty, or $C_{\mathrm{sm}}^H \hookrightarrow C$ is an effective Cartier divisor finite étale over S. Moreover, C_{sm}^H commutes with arbitrary base change.*

Proof. Let $h \in H$ be a generator. For any map $T \to S$, the universal property of equalizers implies that $(C^H)_T$ is the equalizer of $\mathrm{id}, h : C_T \rightrightarrows C_T$. Since taking the smooth locus commutes with arbitrary base change [Stacks, 0C3H], so does C_{sm}^H.

Now suppose C_{sm}^H is non-empty. For any geometric point $\overline{z} \in C_{\mathrm{sm}}^H$, its image in C must land in a marking, so étale locally in C, $C_{\mathrm{sm}}^H \hookrightarrow C$ looks like $\operatorname{Spec} A[t]/\langle(\zeta - 1)t\rangle \to \operatorname{Spec} A[t]$, where A is the strict local ring at the image

[11] If S is reduced, then this property defines $\mathcal{R}_\pi$ as a closed subscheme. In the general case, the topological invariance of the étale site [Stacks, 04DZ] applied to the universal homeomorphism $S_{\mathrm{red}} \to S$ yields the uniqueness up to S-isomorphisms.

[12] This is also the fiber product $C^h = C \times_{(\mathrm{id},h), C \times_S C, \Delta} C$.

of $\overline{z}$ in S, ζ is a primitive $|H|$-th root of unity, and h acts on $A[t]$ linearly in A sending $t \mapsto \zeta t$. This shows that $C_{\mathrm{sm}}^H \hookrightarrow C$ is a closed immersion, and since $|G|$ is invertible on the base, $\zeta - 1 \in A^\times$, so C_{sm}^H is also étale over S. Since $C \to S$ is proper, this implies that C_{sm}^H is moreover finite étale, so it is an effective Cartier divisor as desired. $\square$

Let $H \leq G$ be a non-trivial cyclic subgroup, and let $K \supset H$ be a cyclic subgroup containing H. Then $C_{\mathrm{sm}}^K \subset C_{\mathrm{sm}}^H$ is a closed immersion of finite étale S-schemes, so C_{sm}^K is an open and closed subscheme of C_{sm}^H. Let

$$\Delta(H) := C_{\mathrm{sm}}^H - \bigcup_{K \supsetneq H} C_{\mathrm{sm}}^K,$$

where the union runs over all cyclic subgroups of G properly containing H. Thus, the support of $\Delta(H)$ consists precisely of the points $x \in C_{\mathrm{sm}}$ such that

(a) $Hx = x$,
(b) H acts trivially on the residue field $\kappa(x)$, and
(c) no strictly larger subgroup $K \supsetneq H$ satisfies (a) and (b).

It follows from this description that $\Delta(H) \cap \Delta(K) = \emptyset$ if $H, K \leq G$ are distinct non-trivial cyclic subgroups.

Definition 2.2.2. Let $\pi : C \to D$ be an admissible G-cover of a stable n-pointed curve D over S. The *reduced ramification divisor* is the divisor

$$\mathcal{R}_\pi := \bigsqcup_{H \leq G} \Delta(H),$$

where H runs over all non-trivial cyclic subgroups of G.

PROPOSITION 2.2.3. *Let $\pi : C \to D$ be an admissible G-cover of a stable n-pointed curve $(D, \sigma_1, \ldots, \sigma_n)$ over S. Suppose S is connected, and let e_i denote the ramification index of any point $x \in \pi^{-1}(\sigma_i)$. Then the reduced ramification divisor $\mathcal{R}_\pi \subset C$ is an effective Cartier divisor finite étale over S, supported on the set of non-étale points of $C_{\mathrm{sm}} \to D_{\mathrm{sm}}$. If $e_i \geq 2$, then $\sigma_i^* \mathcal{R}_\pi$ is finite étale over S of degree $|G|/e_i$.*

Proof. The reduced ramification divisor $\mathcal{R}_\pi$ is a disjoint union of finite étale S-schemes, so it is also finite étale over S, hence an effective Cartier divisor. The description of the degree and support follows from the description of $\Delta(H)$ and the local picture above a marking. $\square$

Remark 2.2.4. The "reduced ramification divisor" is only reduced in the relative sense that it is unramified over S. It follows from the étale local picture that it is reduced in the usual "absolute" sense if and only if S is reduced. We call it the reduced ramification divisor to avoid confusion with the ramification divisor that appears in the proof of the Riemann-Hurwitz formula, which can be defined as

$$\mathfrak{R}_\pi := \mathrm{Div}(\pi^* \omega_{D/S} \to \omega_{C/S}),$$

where ω denotes the relative dualizing sheaf, and Div is taken in the sense of Knudsen-Mumford [KM76, §2]. The restriction of $\mathfrak{R}_\pi$ to the preimage of a marking is a multiple of $\mathcal{R}_\pi$. In the notation of Proposition 2.2.3, we have $\sigma_i^*\mathfrak{R}_\pi = (e_i - 1)\sigma_i^*\mathcal{R}_\pi$ [BR11, §4.1.2].

Let $\pi : C \to D$ be an admissible G-cover of a stable n-pointed curve $(D, \sigma_1, \ldots, \sigma_n)$ over S. If $H \leq G$ is a cyclic subgroup, then $\Delta(H)^g = \Delta(g^{-1}Hg)$ for any $g \in G$, so the action of G on C restricts to an action on $\mathcal{R}_\pi$ which is transitive on fibers over D. Since $\mathcal{R}_\pi/S$ is finite étale, we may study the structure of $\mathcal{R}_\pi$ via Galois theory:

Proposition 2.2.5. *Let $\pi : C \to D$ be an admissible G-cover of a prestable pointed curve (D, σ) over (a $\mathbb{Z}[1/|G|]$-scheme) S with ramification indices e above σ. Let $\overline{s}$ be a geometric point of S, and let $\overline{x}$ be a geometric point of C lying over $\sigma(\overline{s})$. Then*

(a) *the connected components of $\mathcal{R}_\pi$ are all isomorphic, and the connected component $R \subset \mathcal{R}_\pi$ containing $\overline{x}$ is finite étale Galois over S with Galois group a subgroup of $N_G(G_{\overline{x}})/G_{\overline{x}}$, where $N_G(G_{\overline{x}})$ denotes the normalizer of $G_{\overline{x}}$ inside G;*
(b) *if, moreover, S is regular and integral and $\Gamma(S, \mathcal{O}_S)$ contains a primitive e-th root of unity, where e denotes the common ramification indices of π above σ, then $\mathrm{Gal}(R/S)$ is isomorphic to a subgroup of $C_G(G_{\overline{x}})/G_{\overline{x}}$, where C_G denotes the centralizer.*

Remark 2.2.6. By working universally (see Proposition 3.2.2), the conditions that S be regular and integral in part (b) of the proposition can be removed.

Proof. Let $\Pi := \pi_1(S, \overline{s})$. Then by Galois theory we have commuting actions of Π and G on the geometric fiber $F := (\mathcal{R}_\pi)_{\overline{s}}$. Thus G acts on the set of Π-orbits of F, and for any $z \in F$, the decomposition group $\mathbb{D} := \mathrm{Stab}_G(\Pi \cdot z)$ acts transitively on the orbit $\Pi \cdot z$. It follows that the inertia group G_z acts trivially on $\Pi \cdot z$, so G_z is normal in $\mathrm{Stab}_G(\Pi \cdot z)$, and the connected component $R \subset \mathcal{R}_\pi$ corresponding to $\Pi \cdot z$ is Galois over S with Galois group $\mathbb{D}/G_z \leq N_G(G_z)/G_z$. This proves (a).

Now suppose in addition that S is regular and integral and contains a primitive e-th root of unity. Let $\eta \in S$ be the generic point. Since R/S is étale, R is irreducible. Let $\epsilon \in R$ be the unique generic point lying over η, and let $\overline{\epsilon}$ be a geometric point mapping to ϵ. Then we find that $\mathbb{D} = G_\epsilon := \mathrm{Stab}_G(\epsilon)$, and taking $\overline{s} = \overline{\eta}$ in the above discussion, it remains to show that $G_{\overline{\epsilon}}$ is contained in the center $Z(\mathbb{D})$ of $\mathbb{D}$. Let $A := \widehat{\mathcal{O}_{D_\eta, \sigma(\eta)}}$ be the complete local ring of the generic fiber D_η at the branch point $\sigma(\eta)$, and let $K := \mathrm{Frac}(A)$. Let $B := \widehat{\mathcal{O}_{C_\eta, \epsilon}}$, and let $L := \mathrm{Frac}(B)$. Then L/K is Galois with Galois group $\mathbb{D} =$

G_ϵ and inertia group $G_{\overline{\epsilon}}$. The vector space $\mathfrak{m}_B/\mathfrak{m}_B^2$ is a 1-dimensional vector space over $B/\mathfrak{m}_B$, and since we are in the tame case, the local representation $\chi_{\overline{\epsilon}} : G_{\overline{\epsilon}} \to \mathrm{GL}(\mathfrak{m}_B/\mathfrak{m}_B^2)$ is faithful [Ser79, §IV.2]. If $g \in \mathbb{D}$ and $\gamma \in G_{\overline{\epsilon}}$, then since by assumption $\Gamma(S, \mathcal{O}_S)$ contains all e-th roots of unity, for any $v \in \mathfrak{m}_B/\mathfrak{m}_B^2$, we have $g^{-1}\gamma g v = g^{-1}(\zeta \cdot gv) = \zeta v$ for some e-th root of unity ζ. Thus $g^{-1}\gamma g$ also acts by multiplication by ζ, but since $\chi_{\overline{\epsilon}}$ is faithful, this implies that $g^{-1}\gamma g = \gamma$ for all $g \in G$, so $G_{\overline{\epsilon}} \le Z(\mathbb{D})$, which proves (b). $\square$

2.3. *The Higman invariant*. Let G be a finite group. The stacks $\mathcal{A}dm(G)$ are typically not geometrically connected. Different connected components can often be distinguished by a natural combinatorial invariant called the Higman invariant, which analytically over $\mathbb{C}$ is the conjugacy class in G given by the monodromy of a small positively oriented loop winding once around the branch point. When G is a matrix group, the trace of this class is called the *trace invariant* (see Section 5.1). Here we discuss the Higman invariant over algebraically closed fields.

Let (D, O) be a 1-pointed prestable curve over an algebraically closed field k. Let G be a finite group of order invertible in k, and let $\pi : C \to D$ be an admissible G-cover. If $k = \mathbb{C}$, then for any base point $y \in D_{\mathrm{gen}}(\mathbb{C})$ and any point $x \in \pi^{-1}(y)$, for $\gamma \in \pi_1^{\mathrm{top}}(D_{\mathrm{gen}}(\mathbb{C}), y)$, let $\gamma \cdot x$ denote the endpoint of the unique lift of γ to C which starts at x. The monodromy representation at x is the unique homomorphism

$$\varphi_x : \pi_1^{\mathrm{top}}(D_{\mathrm{gen}}(\mathbb{C}), y) \longrightarrow G$$

satisfying $\gamma \cdot x = x \cdot \varphi_x(\gamma)$ for all $\gamma \in \pi_1^{\mathrm{top}}(D_{\mathrm{gen}}(\mathbb{C}), y)$.

Varying the choice of $x \in \pi^{-1}(y)$ amounts to post-composing φ_x with an inner automorphism of G. Thus, if $\gamma_O \in \pi_1^{\mathrm{top}}(D_{\mathrm{gen}}(\mathbb{C}), y)$ is a small loop winding once counter-clockwise around O, then $\{\varphi_x(\gamma_O) \mid x \in \pi^{-1}(y)\}$ is a conjugacy class of G, which we call the (*topological*) *Higman invariant* of π (at $O \in D$).

Now suppose k is a general algebraically closed field. Let $x \in \pi^{-1}(O)$, and let e be its ramification index. Since π is admissible, x is a smooth point of C. Let T_x^* denote its cotangent space. Then the stabilizer $G_x := \mathrm{Stab}_G(x)$ is cyclic of order e, and the right action of G on C defines a faithful *local representation*

$$\chi_x : G_x \to \mathrm{GL}(T_x^*) = k^\times \tag{2.1}$$

so it gives an isomorphism $\chi_x : G_x \xrightarrow{\sim} \mu_e(k)$. If $x' \in \pi^{-1}(O)$ is another point, then $x' = xg$ for some $g \in G$, so the conjugation $i_{g^{-1}} : h \mapsto g^{-1}hg$ induces an isomorphism $i_{g^{-1}} : G_x \xrightarrow{\sim} G_{xg}$. Computing on G-invariant open affines, we find that the following diagram is commutative:

$$
\begin{tikzcd}
G_x \arrow[r, "\chi_x"] \arrow[d, "i_{g^{-1}}"] & \mathrm{GL}(T_x^*) \arrow[d, equal] \\
G_{xg} \arrow[r, "\chi_{xg}"] & \mathrm{GL}(T_{xg}^*).
\end{tikzcd}
\tag{2.2}
$$

Thus, if $\zeta_e \in k^\times$ is a primitive e-th root of unity, then the conjugacy class of $\chi_x^{-1}(\zeta_e)$ in G is independent of the choice of $x \in \pi^{-1}(O)$.

Definition 2.3.1. Let k be an algebraically closed field, and let (D, O) be a 1-pointed prestable curve over k. Let G be a finite group of order invertible in k, and let $\pi : C \to D$ be an admissible G-cover. Let e be the common ramification indices of π at points above $O \in D$. If $\zeta_n \in k^\times$ is a primitive n-th root of unity where $e \mid n$, then using the notation above, the (*algebraic*) *Higman invariant of* π *relative to* ζ_n is the conjugacy class of $\chi_x^{-1}(\zeta_n^{n/e})$ for any $x \in \pi^{-1}(O)$, denoted $\mathrm{Hig}_{\zeta_n}(\pi)$. By (2.2), this is independent of the choice of $x \in \pi^{-1}(O)$.

Now suppose our universal base scheme $\mathbb{S}$ is such that there exists a primitive n-th root of unity $\zeta_n \in \Gamma(\mathbb{S}, \mathcal{O}_{\mathbb{S}})$. Let $\xi : \operatorname{Spec} \Omega \to \mathcal{A}dm(G)$ be a geometric point, corresponding to an admissible G-cover $\pi_\xi : C \to E$ over Ω, where (E, O) is a 1-generalized elliptic curve. Thus, to any geometric point ξ, we may associate an integer $e(\xi)$ (the ramification index of the corresponding admissible G-cover). If $e(\xi) \mid n$, then relative to our choice of ζ_n, we may also associate the conjugacy class

$$\mathrm{Hig}_{\zeta_n}(\xi) := \mathrm{Hig}_{\zeta_n}(\pi_\xi).$$

Note that $e(\xi)$ is also just the order of (any representative of) the Higman invariant. Let $|\mathcal{A}dm(G)|$ denote the underlying topological space of the stack $\mathcal{A}dm(G)$ [Stacks, 04XE]. Since $e(\xi)$ and $\mathrm{Hig}_{\zeta_n}(\xi)$ are invariant under field extensions $\Omega \subset \Omega'$, they define functions on $|\mathcal{A}dm(G)|$.

Proposition 2.3.2. *Let* $\mathrm{Cl}(G)$ *denote the set of conjugacy classes of* G. *The functions*

$$e : |\mathcal{A}dm(G)| \longrightarrow \mathbb{N},$$
$$\mathrm{Hig}_{\zeta_n} : |\mathcal{A}dm(G)| \longrightarrow \mathrm{Cl}(G)$$

are locally constant.

Proof. The statement for e is evident from the definition of admissible G-covers. The proof for Hig_{ζ_n} is identical to the proof of [BBCL22, Prop. 2.5.1]. In the language of stable marked G-curves, this also follows from [BR11, Prop. 3.2.5], using the equivalence of Theorem 2.4.9. $\square$

Remark 2.3.3. Over $k = \mathbb{C}$, the (topological) Higman invariant of the analytification of π agrees with the (algebraic) Higman invariant of π relative to $\zeta_n = \exp(\frac{2\pi i}{n})$. This is equivalent to saying that monodromy around a branch point induces multiplication by $\exp(\frac{2\pi i}{n})$ on cotangent (equivalently, tangent) spaces. It will be useful to keep in mind that if E is an elliptic curve over $\mathbb{C}$ with origin O, $E^\circ := E - O$, $x_0 \in E^\circ(\mathbb{C})$, and $a, b \in \pi_1^{\text{top}}(E^\circ(\mathbb{C}), x_0)$ is a basis for the fundamental group with positive intersection number (a "positively oriented basis"), then the conjugacy class of the commutator $[b, a] \in \pi_1^{\text{top}}(E^\circ(\mathbb{C}), x_0)$ is represented by a positively oriented loop in $E^\circ(\mathbb{C})$ winding once around the puncture. Thus, if $\pi : C \to E$ is an admissible G-cover, $x_0 \in E^\circ(\mathbb{C})$ and $x \in \pi^{-1}(x_0)$ with associated monodromy representation $\varphi_x : \pi_1^{\text{top}}(E^\circ(\mathbb{C}), x_0) \to G$, then φ_x is surjective and the Higman invariant of π is the conjugacy class of $\varphi_x([b, a]) = [\varphi_x(b), \varphi_x(a)]$. In particular, the Higman invariant of an admissible G-cover can always be expressed as a commutator of a generating pair of G. This implies, for example, that abelian G-covers of elliptic curves unramified away from the origin are in fact unramified everywhere, or equivalently, if G is abelian, then objects of $\mathcal{A}dm(G)$ have trivial Higman invariant.

Definition 2.3.4. In light of the remark, when $\mathbb{S} = \operatorname{Spec} \overline{\mathbb{Q}}$, we will always consider the Higman invariant relative to $\exp(2\pi i/|G|)$. In this case, for a geometric point $\xi : \operatorname{Spec} \Omega \to \overline{\mathcal{M}(G)}$, we will simply write

$$\text{Hig}(\xi) := \text{Hig}_{\exp(2\pi i/|G|)}(\xi).$$

From the local constancy of the Higman invariant, it follows that we have decompositions

$$\overline{\mathcal{M}(G)}_{\overline{\mathbb{Q}}} = \bigsqcup_{\mathfrak{c} \in \text{Cl}(G)} \overline{\mathcal{M}(G)}_{\mathfrak{c}} \quad \text{and} \quad \mathcal{A}dm(G)_{\overline{\mathbb{Q}}} = \bigsqcup_{\mathfrak{c} \in \text{Cl}(G)} \mathcal{A}dm(G)_{\mathfrak{c}},$$

where $\mathcal{M}(G)_{\mathfrak{c}} \subset \mathcal{M}(G)_{\overline{\mathbb{Q}}}$ (resp. $\mathcal{A}dm(G)_{\mathfrak{c}} \subset \mathcal{A}dm(G)_{\overline{\mathbb{Q}}}$) is the open and closed substack consisting of objects with Higman invariant $\mathfrak{c}$. Let $\overline{M(G)}_{\mathfrak{c}}, Adm(G)_{\mathfrak{c}}$ denote their coarse schemes.

2.4. *Comparison with stable marked G-curves.* An admissible G-cover is a map $\pi : C \to D$ satisfying certain properties. An alternative approach is to forget D and only remember the curve C together with its G-action and a suitable marking divisor $R \subset C$. We will see that π induces an isomorphism $C/G \cong D$, so nothing is lost in this approach. Moreover this perspective will be convenient later when we describe the deformation theory for admissible G-covers in Proposition 2.5.3 below. In this section we make precise the relationship between these two viewpoints. The results here are not new and can be viewed as an exposition of [ACV03, App. B]. However our terminology here follows Bertin-Romagny [BR11, Def. 4.3.4]. We begin with a well-known lemma.

LEMMA 2.4.1. *Let $(C/S, R)$ be a stable marked curve. Let G be a finite group acting S-linearly on C. Then*

(a) *C is a union of G-invariant affine opens.*
(b) *The categorical quotient $\pi : C \to C/G$ exists (as a scheme) and satisfies*
  - *π is finite;*
  - *π is surjective, the fibers of π are G-orbits, and C/G has the quotient topology;*
  - *π induces an isomorphism $\mathcal{O}_{C/G} \xrightarrow{\sim} (\pi_*\mathcal{O}_C)^G$; and*
  - *for every $x \in C$ with image $s \in S$, the residue field $\kappa(x)$ is a normal extension of $\kappa(s)$, and the stabilizer G_x surjects onto $\mathrm{Aut}(\kappa(x)/\kappa(s))$.*

(c) *The quotient C/G commutes with arbitrary base change.*

Proof. If C satisfies (a), then (b) follows from [GR71, Exp. V, Cor. 1.5, Prop. 1.8], and (c) follows from [KM85, Prop. A7.1.3(4)] (using our standing tameness assumption). It remains to show (a). This is Zariski local on the base, so we may assume S affine. Recall that a scheme X satisfies property (AF) if any finite set of points is contained in an affine open [Ryd13, App. B].[13] Since $(C/S, R)$ is stable, $\omega_{C/S}(R) := \omega_{C/S} \otimes \mathcal{O}_C(R)$ is ample on C [Stacks, 0D2S] [Knu83, Cor. 1.9] (this uses the affineness of S), so C satisfies (AF) [Gro61, II, Cor. 4.5.4]. Then for any point $x \in C$, let $W \subset C$ be an open affine containing the orbit Gx. Then $\cap_{g \in G} gW$ is a G-invariant open affine neighborhood of x. This implies (a). □

Remark 2.4.2. The conclusions of Lemma 2.4.1 are true more generally for any finite group acting S-linearly on a finite type S-scheme which satisfies property (AF). Note that the tameness assumption on $|G|, S$ was only used part (c).

Definition 2.4.3. A stable marked G-curve is a stable marked curve $(C/S, R)$ equipped with a faithful *right*-action of G satisfying

(a) the G-action preserves the divisor R;
(b) $C \to C/G$ is étale on $C_{\mathrm{sm}} - R$ (equivalently, G acts with trivial inertia on $C_{\mathrm{sm}} - R$); and
(c) the action at every geometric node is *balanced* in the sense of Remark 2.1.5, observation (e).

A morphism of stable marked G-curves is a morphism of the underlying prestable curves which both preserves the marking and is G-equivariant.

[13](AF) is short for *affine finie*. Property (AF) is also sometimes called the *Chevalley-Kleiman property* [Kol12, Def. 47].

Proposition 2.4.4. *Given a stable marked G-curve $(C/S, R)$, the quotient $(C/G, R/G)$ is a stable marked curve and the quotient map $\pi : C \to C/G$ is finite flat.*

Proof. By Noetherian approximation (Remark 6.1.1), we are reduced to the case where S is of finite type over $\mathbb{Z}$. By Lemma 2.4.1, the quotient exists and π is finite surjective. By [GR71, Cor. V.1.5], C/G is separated and of finite presentation over S. The surjectivity of π implies $C/G \to S$ is universally closed, hence proper. Since C/S is flat, by the fiberwise criteria of flatness [Stacks, 039B], to check that π and $C/G \to S$ are flat, it suffices to check that π is flat on fibers over S. By Lemma 2.4.1(c), quotients commute with base change, so it suffices to check stability and flatness in the case $S = \operatorname{Spec} k$ where k is an algebraically closed field. In this case flatness is clear at smooth points; at nodes it follows immediately from the étale-local picture, using the balanced assumption (see Proposition 6.1.4). The same local picture implies that C/G is prestable.

Finally we claim that the quotient $(C/G, R/G)$ is stable. Suppose $Z \subset C/G$ is a non-stable component, with normalization Z'. Then Z' has genus at most 1. If Z' has genus 1, then we must have $Z = Z'$ and it must have no nodes or markings, but this implies that if W is any irreducible component of the preimage of Z in C, then W contains no nodes or markings, so $W \to Z$ is étale, so by Riemann-Hurwitz, W also has genus 1, so W is also unstable. Now suppose $Z' \cong \mathbb{P}^1$ has genus 0, and let $W \subset C$ be an irreducible component mapping to Z, with normalization W'. Then $W' \to Z'$ is étale away from the complement of two points, so $W' \to Z'$ is a totally ramified cyclic cover, so W' is also genus 0 with at most two special points, so W is also unstable. $\square$

Definition 2.4.5. Let $\pi : C \to D$ be an admissible G-cover of an n-pointed stable curve $(D, \{\sigma_i\}_{1 \le i \le n})$. Let $\mathcal{R}_\pi$ be the reduced ramification divisor, and let $J \subset \{1, \dots, n\}$ be the subset of indices j such that $\mathcal{R}_\pi$ does not meet the divisor $Z_j := C \times_{D,\sigma_j} S$. For each i, let

$$R_i := \begin{cases} Z_i & \text{if } i \in J, \\ \mathcal{R}_\pi \times_{D,\sigma_i} S & \text{if } i \notin J. \end{cases}$$

We call $\sqcup_{i=1}^n R_i$ the *marking associated to the admissible G-cover π*.

Proposition 2.4.6. *Let $\pi : C \to D$ be an admissible G-cover of an n-pointed stable curve $(D, \{\sigma_i\}_{1 \le i \le n})$, and let $R := \sqcup_{i=1}^n R_i$ be the associated marking as in Definition 2.4.5. Then $(C/S, R)$ is a stable marked G-curve. The categorical quotient C/G exists and commutes with arbitrary base change, and π induces an isomorphism $C/G \xrightarrow{\sim} D$.*

Proof. Since π is admissible, the G-action is balanced on the nodes. Since π is étale above D_{gen}, and the support of $\mathcal{R}_\pi$ is precisely the set of points in C_{sm} at which π is not étale, the image of $\mathcal{R}_\pi$ in D is contained in the marking divisor and $\sqcup_{j\in J} Z_j$ is étale over S. Thus $R \subset C$ is an effective Cartier divisor, finite étale over S. To show that $(C/S, R)$ is stable marked, we may assume $S = \operatorname{Spec} k$ for k an algebraically closed field. Let $Z \subset D$ be an irreducible component with normalization Z', and let $W \subset C$ be an irreducible component mapping to Z, with normalization W'. We say that a point of W' (resp. Z') is special if it maps to a node or marking of W (resp. Z). By Riemann-Hurwitz, the stability of W is clear if Q' has genus ≥ 2, and since a point of W is special if and only if it maps to a special point of Z (Remark 2.1.5(a)), stability is also clear if Z' contains at least three special points. The only remaining case is when Z' has genus 1, containing at least one special point, but again in this case we find W' has genus at least 1 with at least one node or marking, so W' is also stable.

By Lemma 2.4.1, the categorical quotient C/G exists, commutes with arbitrary base change, and is defined affine locally by taking G-invariants. Then π factors uniquely through a finite map $\alpha : C/G \to D$ which by Definition 2.1.4(3) must be an isomorphism over D_{gen}. It follows from the local picture at the nodes and markings that α is an isomorphism there as well. □

Definition 2.4.7. Let $\overline{\mathcal{H}}_{g,n,G}$ denote the category whose objects are stable marked G-curves $(C/S, R)$ equipped with a decomposition $R = \bigsqcup_{i=1}^n R_i$ into open and closed subschemes such that

(1) G preserves each R_i,

(2) the map $R_i/G \to S$ is an isomorphism[14] for each i, and

(3) C/G is a prestable curve of genus g,

and whose morphisms are fiber squares, preserving the decomposition $R = \bigsqcup_{i=1}^n R_i$. Then $\overline{\mathcal{H}}_{g,n,G}$ is a category fibered in groupoids over $\underline{\mathbf{Sch}}/\mathbb{S}$. Let $\mathcal{H}_G \subset \overline{\mathcal{H}}_G$ denote the subcategory consisting of pairs $(C/S, R)$ where C/S is smooth. Let $\overline{\mathcal{H}}_G := \overline{\mathcal{H}}_{1,1,G}$, and similarly let $\mathcal{H}_G := \mathcal{H}_{1,1,G}$.

PROPOSITION 2.4.8. *Let $(C/S, R = \bigsqcup_{i=1}^n R_i)$ be an object of $\overline{\mathcal{H}}_{g,n,G}$. Let σ_i denote the section of C/G determined by R_i/G. Then $(C/G, \{\sigma_i\}_{1\leq i\leq n})$ is a stable n-pointed curve and the quotient map $\pi : C \to C/G$ is an admissible G-cover.*

Proof. By Proposition 2.4.4, $(C/G, \{\sigma_i\})$ is a stable n-pointed curve. It remains to check that π is an admissible G-cover. The most difficult thing to check is that π has the correct local picture at the nodes and markings and

[14] Equivalently, G acts transitively on the geometric fibers of $R_i \to S$.

that the G-action is balanced at the nodes. This is done in Proposition 6.1.4 in the appendix. This local picture then implies that π maps nodes to nodes. Since G acts without inertia on $C_{\mathrm{sm}} - R$, π is a G-torsor above $(C/G)_{\mathrm{gen}}$, so π is an admissible G-cover as desired. $\square$

THEOREM 2.4.9. *For $g, n \in \mathbb{Z}_{\geq 0}$, the map Φ sending an object $(C/S, R = \sqcup_{i=1}^n R_i)$ in $\overline{\mathcal{H}}_{g,n,G}$ to the admissible G-cover $C \to C/G$ together with the n sections of C/G determined by R_i/G gives an equivalence of categories*

$$\Phi : \overline{\mathcal{H}}_{g,n,G} \xrightarrow{\sim} \mathcal{A}dm_{g,n}^{\mathrm{conn}}(G).$$

A quasi-inverse is given by the functor Ψ sending an admissible G-cover $(C \xrightarrow{\pi} D, \{\sigma_i : S \to D\}_{1 \leq i \leq n})$ to $(C/S, R = \sqcup_{i=1}^n R_i)$, where here R_i is as in Definition 2.4.5. In particular, $\overline{\mathcal{H}}_G$ is a smooth proper Deligne-Mumford stack of pure dimension 1 over $\mathbb{S}$.

Proof. Since any admissible G-cover $C \to D$ is a quotient map (Proposition 2.4.6), Φ is fully faithful, so maps in $\mathcal{A}dm_{g,n}^{\mathrm{conn}}(G)$ are precisely given by G-equivariant maps of the covering curve preserving the decomposition of the marking $R = \sqcup_{i=1}^n R_i$. It is essentially surjective since $\Psi \circ \Phi$ is equal to the identity functor on $\overline{\mathcal{H}}_{g,n,G}$. $\square$

2.5. *Relation to the moduli stack of elliptic curves with G-structures*. In this section we compare $\mathcal{A}dm(G)$ (equivalently $\overline{\mathcal{H}}_G$) to the moduli stack $\mathcal{M}(G)$ of elliptic curves with G-structures [Che18], [Ols12], [PdJ95], [DM69]. The main result is that the open substack $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ corresponding to smooth covers is an étale gerbe over $\mathcal{M}(G)$, and $\mathcal{M}(G)$ can be obtained from $\mathcal{A}dm^0(G)$ by a rigidification process removing $Z(G)$ from all the automorphism groups of objects in $\mathcal{A}dm^0(G)$. The same process applied to $\mathcal{A}dm(G)$ results in a smooth compactification of $\mathcal{M}(G)$, which we denote $\overline{\mathcal{M}(G)}$. In this section we explain these relationships, following [ACV03], [Che18], and [BR11].

Let G be a finite group. A G-torsor over a scheme X is a finite étale morphism $p : Y \to X$ together with an X-linear right action of G on Y such that p acts freely and transitively on geometric fibers. A morphism of G-torsors over X is a G-equivariant morphism over X. For an elliptic curve E/S with zero divisor O, recall that $E^\circ := E - O$. Let $\mathcal{T}_G^{\mathrm{pre}}$ denote the presheaf

(2.3)
$$\begin{aligned} \mathcal{T}_G^{\mathrm{pre}} : \mathcal{M}(1) &\longrightarrow \underline{\mathbf{Sets}} \\ E/S &\mapsto \{G\text{-torsors } X \to E^\circ \text{ with geom. connected fibers over } S\}/\cong . \end{aligned}$$

Let $\mathcal{T}_G$ be the sheafification of $\mathcal{T}_G^{\mathrm{pre}}$ with respect to the topology on $\mathcal{M}(1)$ inherited from $(\underline{\mathbf{Sch}}/\mathbb{S})_{\acute{e}t}$. Then $\mathcal{M}(G)$ is the stack associated to $\mathcal{T}_G$. For an elliptic curve E/S, a G-structure[15] on E/S is by definition an element of the set $\mathcal{T}_G(E/S)$.

If α, α' are isomorphism classes of G-torsors on E/S, E'/S', then a morphism $(E'/S', \alpha') \to (E/S, \alpha)$ thus consists of a cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

such that $f^*\alpha = \alpha'$. Thus, morphisms in $\mathcal{M}(G)$ are determined by morphisms of elliptic curves, so the map $\mathcal{M}(G) \to \mathcal{M}(1)$ is representable. Using the relative fundamental group, $\mathcal{T}_G$ is checked to be finite locally constant on $\mathcal{M}(1)$, and hence $\mathcal{M}(G) \to \mathcal{M}(1)$ is finite étale. In particular, it is a smooth Deligne-Mumford stack.[16]

Remark 2.5.1. We make some remarks.

(a) If $S = \operatorname{Spec} k$ with k a separably closed field, and E/k is an elliptic curve, then the stalk of $\mathcal{T}_G^{\mathrm{pre}}$ (resp. $\mathcal{T}_G$) at E/k is precisely $\mathcal{T}_G^{\mathrm{pre}}(E/k)$ (resp. $\mathcal{T}_G(E/k)$) [Stacks, 06VW]. Thus, since sheafification preserves stalks [Stacks, 00Y8], $\mathcal{T}_G^{\mathrm{pre}}(E/k) = \mathcal{T}_G(E/k)$ is precisely the set of isomorphism classes of geometrically connected G-torsors over E°/k.

(b) If E/S is an elliptic curve, then an object of $\mathcal{T}_G(E/S)$ is given by an étale covering $\{S_i \to S\}$, and G-torsors X_i on each $E_i^\circ := E^\circ \times_S S_i$ with geometrically connected fibers over S_i whose common "overlaps" are isomorphic. However, there is no requirement that one can choose the isomorphisms to satisfy a cocycle condition, and hence the G-torsors $\{X_i\}$ need not glue to give a G-torsor on E°. If G has trivial center, then such G-torsors have trivial automorphism groups, and hence the cocycle condition is automatic, so in this case we have $\mathcal{T}_G^{\mathrm{pre}} = \mathcal{T}_G$ [Che18, Prop. 2.2.6(3)].

(c) Given an elliptic curve E/S, a G-torsor on E° with geometrically connected S-fibers defines a G-structure on E/S. This gives a map from the set of isomorphism classes of geometrically connected G-torsors on E°/S

[15] This G-structure was originally called a *Teichmüller structure of level G* in [DM69, §5]. In [Che18], G-structures were studied as non-abelian analogues of the usual congruence level structures attached to elliptic curves. While the definition of G-structures here differs from the one used in [Che18], the resulting objects are isomorphic. Namely, taking monodromy representations defines a map from $\mathcal{T}_G^{\mathrm{pre}}$ to the presheaf of [Che18, Def. 2.2.3] which is locally an isomorphism. Thus, their sheafifications are isomorphic.

[16] As defined, $\mathcal{M}(G)$ is an algebraic stack for the étale topology. By [Stacks, 076U], it is also an algebraic stack for the fppf topology (i.e., an algebraic stack in the sense of the stacks project).

to the set of G-structures $\mathcal{T}_G(E/S)$. As we saw above, this map is a bijection if either $S = \operatorname{Spec} k$ with k a separably closed field, or if G has trivial center, but in general it need not be injective or surjective.[17] While this may make G-structures seem like a somewhat unnatural gadget, the upshot is that the forgetful map $\mathcal{M}(G) \to \mathcal{M}(1)$ is *finite étale*, and so it can be studied using Galois theory. On the other hand, while $\mathcal{A}dm(G)$ is in some sense more natural (and carries a universal family of covers), the forgetful map $\mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ is typically not representable, hence typically not finite, even above $\mathcal{M}(1)$ (see Section 2.5.2). In Proposition 2.5.10 we will show that $\mathcal{M}(G)$ is homeomorphic to $\mathcal{A}dm^0(G)$. Since we will be interested in questions of connectedness, we may (and we will) make use of both perspectives.

2.5.1. *Review of G-structures*. Here we recall some of the salient features of the stacks $\mathcal{M}(G)$.

[17]If G is abelian, then the map is surjective but rarely injective: If $\pi : X \to E^\circ$ is a G-torsor geometrically connected over S and E° admits a section σ, then for *any* G-torsor ξ over S, one can "twist" π in a way that the restriction of the resulting G-torsor π_ξ to σ is isomorphic to ξ, but such that the torsors $\{\pi_\xi\}_\xi$ all determine the same G-structure. If G has non-trivial center, then failure of descent implies that this map is typically not surjective.

THEOREM 2.5.2. *Let G be a finite group. Let*

$$\mathfrak{f} : \mathcal{M}(G) \to \mathcal{M}(1)$$

be the forgetful map. We work universally over $\mathbb{S} = \operatorname{Spec} \mathbb{Z}[1/|G|]$.

(1) (*étaleness*). *The category $\mathcal{M}(G)$ is a Noetherian smooth separated Deligne-Mumford stack, and the forgetful functor $\mathfrak{f} : \mathcal{M}(G) \to \mathcal{M}(1)$ is finite étale.*

(2) (*Coarse moduli and ramification*). *$\mathcal{M}(G)$ admits a coarse moduli scheme $M(G)$ which is a normal affine scheme finite over $M(1) \cong \operatorname{Spec} \mathbb{Z}[1/|G|][j]$, and smooth of relative dimension 1 over $\mathbb{Z}[1/|G|]$. Moreover, $M(G)$ is étale over the complement of the sections $j = 0$ and $j = 1728$ in $M(1)$. If either $6 \mid |G|$ or S is a regular Noetherian $\mathbb{Z}[1/|G|]$-scheme, then $M(G) \times_{\mathbb{Z}[1/|G|]} S$ is the coarse moduli scheme of $\mathcal{M}(G) \times_{\mathbb{Z}[1/|G|]} S$, and it is normal.*

(3) (*Combinatorial description of G-structures*). *Let $\mathbb{L}$ be the set of prime divisors of $|G|$. For any profinite group π, let $\pi^{\mathbb{L}}$ denote the maximal pro-$\mathbb{L}$-quotient of π. Let E be an elliptic curve over S. Let $\overline{x} \in E^\circ$ be a geometric point, and let $\overline{s} \in S$ be the image of $\overline{x}$. The sequence $E^\circ_{\overline{s}} \hookrightarrow E \to S$ induces an outer representation*

$$\rho_{E,\overline{x}} : \pi_1(S, \overline{s}) \to \operatorname{Out}(\pi_1^{\mathbb{L}}(E^\circ_{\overline{s}}, \overline{x}))$$

from which we obtain a natural right action of $\pi_1(S, \overline{s})$ on the set

$$\operatorname{Epi}^{\text{ext}}(\pi_1^{\mathbb{L}}(E^\circ_{\overline{s}}, \overline{x}), G) := \operatorname{Epi}(\pi_1^{\mathbb{L}}(E^\circ_{\overline{s}}, \overline{x}), G)/\operatorname{Inn}(G)$$

of surjective morphisms $\pi_1^{\mathbb{L}}(E^\circ_{\overline{s}}, \overline{x}) \to G$ up to conjugation in G. By Galois theory this action corresponds to a finite étale morphism $F \to S$, which fits into a cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathcal{M}(G) \arrow[d, "\mathfrak{f}"] \\ S \arrow[r, "E/S"] & \mathcal{M}(1). \end{tikzcd}$$

In particular, we obtain a bijection

$$\mathcal{T}_G(E/S) \xrightarrow{\sim} \{\varphi \in \operatorname{Epi}^{\text{ext}}(\pi_1(E^\circ_{\overline{s}}, \overline{x}), G) \bigm| \varphi \circ \rho_{E,\overline{x}}(\sigma) = \varphi \text{ for all } \sigma \in \pi_1(S, \overline{s})\},$$

where recall that $\mathcal{T}_G(E/S)$ is the set of G-structures on E/S.

(4) (*Fibers*). *Let E be an elliptic curve over an algebraically closed field k of characteristic not dividing $|G|$, and let $x_0 \in E^\circ(k)$. Let $x_E : \operatorname{Spec} k \to \mathcal{M}(1)$ be the geometric point given by E. The fiber $\mathfrak{f}^{-1}(x_E)$ is in bijection with the set of connected G-torsors on E°. Taking monodromy representations (see Section 2.3) gives a canonical bijection*

$$\mathfrak{f}^{-1}(x_E) \xrightarrow{\sim} \operatorname{Epi}^{\text{ext}}(\pi_1^{\acute{e}t}(E^\circ, x_0), G). \tag{2.4}$$

If E is an elliptic curve over $\mathbb{C}$, then for $x_0 \in E^\circ(\mathbb{C})$, write

$$\Pi := \pi_1^{\text{top}}(E^\circ(\mathbb{C}), x_0).$$

Taking monodromy representations gives a canonical bijection

$$\mathfrak{f}^{-1}(x_E) \xrightarrow{\sim} \mathrm{Epi}^{\mathrm{ext}}(\Pi, G). \tag{2.5}$$

In particular, if G *is not generated by two elements, then* $\mathcal{M}(G)$ *is empty. Let* $a, b \in \Pi$ *be generators. Let* $\gamma_0, \gamma_{1728}, \gamma_\infty, \gamma_{-I} \in \mathrm{Aut}(\Pi)$ *be the automorphisms given by*

$$\begin{array}{rcl} \gamma_0 : (a,b) & \mapsto & (ab^{-1}, a), \\ \gamma_{1728} : (a,b) & \mapsto & (b^{-1}, a), \end{array} \qquad \begin{array}{rcl} \gamma_\infty : (a,b) & \mapsto & (a, ab), \\ \gamma_{-I} : (a,b) & \mapsto & (a^{-1}, b^{-1}). \end{array}$$

Let $f : \overline{M(G)} \to \overline{M(1)}_{\mathbb{C}}$ *be the map induced by* $\mathfrak{f}$, *where* $\overline{M(G)}$ *denotes a smooth compactification of* $M(G)$ *over* $\mathbb{C}$.[18] *If* $j(E) \neq 0, 1728$, *then viewing* $\overline{M(1)}$ *as the projective line with coordinate* j, *there is a bijection*

$$f^{-1}(j(E)) \xrightarrow{\sim} \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/\langle \gamma_{-I} \rangle$$

such that via (2.5), *the map* $\mathfrak{f}^{-1}(x_E) \to f^{-1}(j(E))$ *induced by* $\mathcal{M}(G) \to M(G)$ *is identified with the the canonical projection*

$$\mathrm{Epi}^{\mathrm{ext}}(\Pi, G) \to \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/\langle \gamma_{-I} \rangle.$$

Letting $j(E)$ *approach* $j = 0, 1728, \infty$ *respectively, we also obtain bijections*

$$f^{-1}(0) \cong \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/\langle \gamma_0 \rangle, \quad f^{-1}(1728) \cong \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/\langle \gamma_{1728} \rangle, \ \textit{and}$$

$$f^{-1}(\infty) \cong \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/\langle \gamma_{-I}, \gamma_\infty \rangle$$

(5) (*Monodromy*). *Let* E *be an elliptic curve over* $\mathbb{C}$, $x_0 \in E^\circ(\mathbb{C})$, *and* $\Pi := \pi_1^{\mathrm{top}}(E^\circ(\mathbb{C}), x_0)$. *Let* $x_E : \operatorname{Spec} \mathbb{C} \to \mathcal{M}(1)$ *be the geometric point given by* E. *Then* Π *is a free group of rank* 2, *and the canonical map* $\Pi \to H_1(E, \mathbb{Z})$ *induces an isomorphism* $\Pi/[\Pi,\Pi] \cong H_1(E,\mathbb{Z})$. *Let* Γ_E *denote the orientation-preserving mapping class group of* $E^\circ(\mathbb{C})$, *and let* $\mathrm{Out}^+(\Pi)$ *be the preimage of* $\mathrm{SL}(H_1(E,\mathbb{Z}))$ *under the canonical map*

$$\alpha : \mathrm{Out}(\Pi) \to \mathrm{GL}(H_1(E, \mathbb{Z})).$$

The outer action of Γ_E *on* Π *is faithful and identifies* Γ_E *with* $\mathrm{Out}^+(\Pi)$. *As* α *is an isomorphism, it induces canonical isomorphisms* $\Gamma_E \xrightarrow{\sim} \mathrm{Out}^+(\Pi) \xrightarrow{\sim} \mathrm{SL}(H_1(E,\mathbb{Z}))$. *The image of* $\gamma_{-I} \in \mathrm{Aut}(\Pi)$ (*see* (4)) *in* $\mathrm{SL}(H_1(E,\mathbb{Z}))$ *is central. The analytic theory identifies* Γ_E *with the topological fundamental group of the analytic moduli stack of elliptic curves* $\mathcal{M}(1)^{\mathrm{an}}$, *from which we obtain canonical isomorphisms* $\mathrm{Out}^+(\Pi)^\wedge \xrightarrow{\sim} \Gamma_E^\wedge \xrightarrow{\sim} \pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, E)$ (*where* $^\wedge$ *denotes profinite completion*). *In particular, we have a canonical*

[18] In fact we will see in Proposition 2.5.10(e) below that $M(G)$ even admits a smooth modular compactification over $\mathbb{Z}[1/|G|]$.

map $\mathrm{Out}^+(\Pi) \hookrightarrow \pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, E)$ *with dense image. Relative to this map, the bijection*

$$\mathfrak{f}^{-1}(x_E) \xrightarrow{\sim} \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$$

of (2.4) *is* $\mathrm{Out}^+(\Pi)$*-equivariant. To summarize, we have canonical isomorphisms*

$$\pi_1^{\mathrm{top}}(\mathcal{M}(1)^{\mathrm{an}}, E) \cong \Gamma_E \cong \mathrm{Out}^+(\Pi) \cong \mathrm{SL}(H_1(E, \mathbb{Z}))$$

and

$$\pi_1^{\acute{e}t}(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, E) \cong \pi_1^{\mathrm{top}}(\mathcal{M}(1)^{\mathrm{an}}, E)^\wedge.$$

(6) (*Functoriality*). *Let* $\mathcal{C}$ *denote the category whose objects are finite groups generated by two elements, and whose morphisms are surjective homomorphisms. If* $f : G_1 \twoheadrightarrow G_2$ *is a morphism in* $\mathcal{C}$, *then we obtain a map* $\mathcal{T}_f^{\mathrm{pre}} : \mathcal{T}_{G_1}^{\mathrm{pre}} \to \mathcal{T}_{G_2}^{\mathrm{pre}}$ *defined by sending the* G_1*-torsor* $X^\circ \to E^\circ$ *to the* G_2*-torsor* $X^\circ/\ker(f) \to E^\circ$ *where the* G_2*-action is given by the canonical isomorphism* $G_1/\ker(f) \cong G_2$ *induced by* f. *This induces a map* $\mathcal{T}_f : \mathcal{T}_{G_1} \to \mathcal{T}_{G_2}$, *whence a map*

$$\mathcal{M}(f) : \mathcal{M}(G_1) \to \mathcal{M}(G_2).$$

The maps $\mathcal{M}(f)$ *make the rule sending* $G \in \mathcal{C}$ *to the map* $\mathcal{M}(G) \to \mathcal{M}(1)$ *into an epimorphism-preserving functor from* $\mathcal{C}$ *to the category*[19] *of stacks finite étale over* $\mathcal{M}(1)$. *Let* E, Π, Γ_E *be as in* (4). *Then in terms of the Galois correspondence for covers of* $\mathcal{M}(1)$, *given a surjection* $f : G_1 \to G_2$, *the induced map* $\mathcal{M}(G_1) \to \mathcal{M}(G_2)$ (*of* $\mathbb{Z}[1/|G|]$*-stacks*) *is given by the* Γ_E*-equivariant map of fibers*

$$f_* : \mathrm{Epi}^{\mathrm{ext}}(\Pi, G_1) \to \mathrm{Epi}^{\mathrm{ext}}(\Pi, G_2)$$

obtained by post-composing every surjection with f.

(7) (*Cofinality — Asada's theorem*). *For any stack* $\mathcal{M}$ *finite étale over* $\mathcal{M}(1)_{\overline{\mathbb{Q}}}$, *there is a finite group* G *such that* $\mathcal{M}$ *is dominated by some connected component of* $\mathcal{M}(G)_{\overline{\mathbb{Q}}}$. *In particular, for any smooth projective curve* X *over* $\overline{\mathbb{Q}}$, *there are an open* $U \subset X$, *a component* $\mathcal{N} \subset \mathcal{M}(G)_{\overline{\mathbb{Q}}}$, *and a finite étale morphism* $\mathcal{N} \to U$. *Here we can even arrange that* $\mathcal{N} \to U$ *be Galois, and for the Galois action to be defined over the field of definition of* $\mathcal{M}_X$ *as an element of* $\pi_0(\mathcal{M}(G)_{\overline{\mathbb{Q}}})$.

Proof. Part (1) is [Che18, Prop. 3.1.4]. Everything in (2) except for normality and étaleness is [Che18, Prop. 3.3.4]. The normality of $M(G)$ follows

[19] Here we mean the (1-)category associated to the (2,1)-category. That is, the morphisms in this category are precisely the 2-isomorphism classes of 1-morphisms; cf. [Noo04, §4]. We note that this category is equivalent to the category of finite locally constant sheaves on $\mathcal{M}(1)$ (with respect to the étale topology).

from the fact that $M(G)$ is the quotient of a smooth representable moduli problem by a finite group [Che18, §3.3.3]. Let $U \subset M(1)$ be the complement of $j = 0, 1728$; to see that $M(G)$ is étale over the complement of U, consider a finite étale surjection $\mathcal{M} \to \mathcal{M}(G)$ with $\mathcal{M}$ representable. (We may, for example, take $\mathcal{M}$ to be the product of $\mathcal{M}(G)$ with the moduli stack of elliptic curves with full level p^2 structure for some $p \mid |G|$.) By [KM85, Cor. 8.4.5], the map $\mathcal{M}_U \to U$ is étale, which implies the étaleness of $\mathcal{M}(G)_U \to U$ [Stacks, 02KM].

Part (3) is [Che18, Prop. 2.2.6(1,2)]. The bijections (2.4) and (2.5) of (4) follow from part (3), setting $S = \operatorname{Spec} k$. For the rest of (4), see [BBCL22, Prop. 2.1.2, Cor. 2.1.3].

For (5), a theorem of Nielsen gives that α is an isomorphism [OZ81, Th. 3.1], and the isomorphism $\Gamma_E^\wedge \cong \pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, E)$ follows from the Riemann existence theorem for stacks [Noo05, Th. 20.4]. The rest of (5) is simply an unfolding of definitions. Part (6) is [Che18, Prop. 3.2.8]. All but the final sentence of (7) is Asada's theorem together with Belyi's theorem (see [Che18, Th. 3.4.2], [BER11], [Asa01, §7]). The final sentence is [BBCL22, Th. 2.2.3]. □

2.5.2. *The relation between $\mathcal{A}dm(G)$ and $\mathcal{M}(G)$.* The key relation between $\mathcal{A}dm(G)$ and $\mathcal{M}(G)$ is that the map $\mathcal{A}dm^0(G) \to \mathcal{M}(1)$ is an étale gerbe. First we show that it is étale. This is a problem in deformation theory (see Proposition 6.5.6). In fact, following [BR11, §5], we can even describe the ramification indices at points lying over the "cusp" of $\overline{\mathcal{M}(1)}$ represented by a nodal cubic. For this it will be useful to work with the equivalent stack $\overline{\mathcal{H}}_G$ of stable marked G-curves (Definition 2.4.7).

Proposition 2.5.3 ([BR11, Th. 5.1.5]). *Let k be an algebraically closed field (of characteristic coprime to $|G|$) and $\overline{x} : \operatorname{Spec} k \to \overline{\mathcal{H}}_G$ be a geometric point with image $\overline{y} \in \overline{\mathcal{M}(1)}$. The point $\overline{x}$ corresponds to a stable marked G-curve $(C/k, R)$, and $\overline{y}$ is given by the 1-generalized elliptic curve $E := C/G$ with origin $O = R/G$. The natural map $\overline{\mathcal{H}}_G \to \overline{\mathcal{M}(1)}$ given by taking quotients by G induces a morphism from the deformation functor of $\overline{x}$ to that of $\overline{y}$. Let Λ be the Cohen ring[20] with residue field k. The deformation functors for $\overline{x}, \overline{y}$ in $\overline{\mathcal{H}}_{G,\Lambda}$ and $\overline{\mathcal{M}(1)}_\Lambda$ are prorepresentable, and the induced map on universal deformation rings is given (with respect to suitable coordinates) by*

$$\Lambda[\![T]\!] \longrightarrow \Lambda[\![t]\!],$$
$$T \mapsto t^e,$$

where $e = 1$ if C is smooth, and otherwise e is the order of the stabilizer G_p of any node $p \in C$. In particular, the map $\overline{\mathcal{H}}_G \to \overline{\mathcal{M}(1)}$ is flat and $\mathcal{H}_G \to \mathcal{M}(1)$

[20] If $\operatorname{char}(k) = 0$, then $\Lambda = k$, and otherwise it is the unique [Mat89, Th. 29.2] complete discrete valuation ring with residue field k and maximal ideal $p\Lambda$.

is étale. Moreover, the substack $\mathcal{H}_G \subset \overline{\mathcal{H}}_G$ is open and dense, and the same is true of $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$.

Proof. This statement is a special case of [BR11, Th. 5.1.5]. Here we sketch the argument in our situation. Let $D_{C,G}$ (resp. D_E) denote the deformation functor of C as a stable marked G-curve (resp. of E as a 1-generalized elliptic curve). Since $\overline{\mathcal{H}}_G, \overline{\mathcal{M}(1)}$ are Deligne-Mumford, all deformation functors are prorepresentable, and the universal deformation rings are the completions of the étale local rings of $\overline{x} \in \overline{\mathcal{H}}_{G,\Lambda}$ and $\overline{y} \in \overline{\mathcal{M}(1)}_\Lambda$ (see Proposition 6.5.5). Because $\overline{\mathcal{H}}_{G,\Lambda}$ and $\overline{\mathcal{M}(1)}_\Lambda$ are smooth and 1-dimensional over $\operatorname{Spec}\Lambda$, the universal deformation rings are power series rings in one variable over Λ [Stacks, 0DYL]. Let

$$\pi : C \to E := C/G$$

be the quotient map. Every deformation of C yields by taking quotients a deformation of E, so π induces a map $D_{C,G} \to D_E$. It remains to describe the induced map of universal deformation rings. The deformation theory of C is described by equivariant cohomology (see [BM00, §3] or [BM06, §3]). We briefly recall some definitions. An $(\mathcal{O}_C, G)$-module is a coherent sheaf which locally on an open affine $\operatorname{Spec} A$ is given by an A-module M equipped with a G-action satisfying $g(am) = g(a)g(m)$ for any $g \in G, a \in A, m \in M$. Given an $(\mathcal{O}_C, G)$-module $\mathcal{F}$ on C, let $\pi_*^G(\mathcal{F})$ be the module on E given by $U \mapsto \Gamma(U, \pi_*\mathcal{F})^G$, and let $\Gamma^G(C, \mathcal{F}) := \Gamma(C, \mathcal{F})^G$. Let $H^i_G(C, \mathcal{F}) := R^i\Gamma^G\mathcal{F}$. Let $\mathcal{T}_C := \mathcal{H}om_C(\Omega^1_{C/k}, \mathcal{O}_C)$ be the tangent sheaf. Since π is finite and $|G|$ is invertible in k, π_*^G is exact, and hence we obtain a canonical isomorphism

$$H^1_G(C, \mathcal{T}_C(-\mathcal{R}_\pi)) \cong H^1(E, \pi_*^G\mathcal{T}_C(-\mathcal{R}_\pi)).$$

By a local calculation [BR11, Prop. 4.1.11],[21] we have $\pi_*^G\mathcal{T}_C(-\mathcal{R}_\pi) = \mathcal{T}_E(-O)$ from which we obtain a canonical isomorphism

$$H^1_G(C, \mathcal{T}_C(-\mathcal{R}_\pi)) \cong H^1(E, \mathcal{T}_E(-O)). \tag{2.6}$$

If C (equivalently E) is smooth, $H^1(E, \mathcal{T}_E(-O))$ is the tangent space of D_E (see [Har10, Th. 5.3] for the unmarked case; also see [ACG11, §XI.3]), and $H^1_G(C, \mathcal{T}_C(-\mathcal{R}_\pi))$ is the tangent space of $D_{C,G}$ [BM00, Prop. 3.2.1]. By (2.6) these tangent spaces are isomorphic (and 1-dimensional), though this does not tell us that the map on deformation spaces induced by $C \mapsto C/G$ induces an isomorphism. To check this, one can use the fact that the deformations of E and C can be described explicitly by Čech cohomology (equivariant in the case of C (see [BM00, §3.1] and [Gro57, §5.5]), but the added complication is minimal due to the invertibility of $|G|$ on k). There is a natural map of Čech complexes

[21]See [BR11, §4.1.2] for the definition of points of type I, II, and III. In their notation, $D = C/G$, and by $\theta_D(-\Delta)$ they mean $\theta_D(-B)$.

induced by the map $C \to E$, which (a) induces the isomorphism (2.6), and (b) is easily seen to agree with the map on tangent spaces $D_{C,G}(k[\epsilon]) \to D_E(k[\epsilon])$ induced by $C \mapsto E$. This shows that $\mathcal{H}_{G,\Lambda} \to \mathcal{M}(1)_\Lambda$ induces an isomorphism of deformation rings and hence is étale by Proposition 6.5.6. Since $\mathcal{H}_G \to \mathcal{M}(1)$ is flat (Theorem 2.1.11(b)), this implies that $\mathcal{H}_G \to \mathcal{M}(1)$ is étale.

Now suppose C is nodal. Let $\mathcal{C}_\Lambda$ be the category of Artin local Λ-algebras. If $x \in C$ is a node with stabilizer G_x, then the local deformation functor $D_{C,G_x,x}$ of C at x [BM06, §3] is given by sending an object $A \in \mathcal{C}_\Lambda$ to the set of deformations of $\widehat{\mathcal{O}_{C,x}} \cong k[\![u,v]\!]/(uv)$ over A as an A-algebra with G_x-action. The natural "localization" morphism $D_{C,G} \to D_{C,G_x,x}$ is *smooth*. (See [BM06, Th. 4.3] for the unmarked case.) The map on tangent spaces $D_{C,G}(k[\epsilon]) \to D_{C,G_x,x}(k[\epsilon])$ can be identified with the map

$$\varphi : \mathrm{Ext}^1_{\mathcal{O}_C,G}(\Omega^1_{C/k}, \mathcal{O}_C(-\mathcal{R}_\pi)) \longrightarrow \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,x},G_x}(\widehat{\Omega}_{\widehat{\mathcal{O}}_{C,x}/k}, \widehat{\mathcal{O}}_{C,x})$$

coming from the local-to-global (equivariant) Ext spectral sequence, which is surjective with kernel $H^1_G(C, \mathcal{T}_C(-\mathcal{R}_\pi)) = H^1(E, \mathcal{T}_E(-O)) = 0$ (see [BM06, Lemme 4.1] for the unmarked case), so the map $D_{C,G} \to D_{C,G_x,x}$ induces an isomorphism on tangent spaces.

Since G is invertible in k, $\mathrm{Ext}^i_{\widehat{\mathcal{O}}_{C,x},G_x}(-,-) = \mathrm{Ext}^i_{\widehat{\mathcal{O}}_{C,x}}(-,-)^G$ for all $i \geq 0$. The fact that the G-action is balanced at x implies that the G-action on $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,x}}(\widehat{\Omega}_{\widehat{\mathcal{O}}_{C,x}/k}, \widehat{\mathcal{O}}_{C,x})$ is trivial (see [BM06, §5.2] and [BR11, Th. 5.1.1]). On the other hand, this latter group is also the tangent space of the usual (non-equivariant) local deformation functor of a node. In fact, if $D_{C,x}$ denotes the usual deformation functor of the node x without G_x-action, then it can be shown using the results of [ST18] that the forgetful map $D_{C,G_x,x} \to D_{C,x}$ is an *isomorphism*, and that a miniversal family for $D_{C,G_x,x}$ is given by $\Lambda[\![U,V,T]\!]/(UV-T)$ with G_x action given by $gU = \chi(g)U, gV = \chi(g)^{-1}V$ for some primitive character $\chi : G_x \to k^\times$. In particular, $\Lambda[\![U,V,T]\!]/(UV-T)$ (without G_x-action) defines a miniversal family for $D_{C,x}$.

Let $R_{C,G_x,x} = \Lambda[\![t]\!]$ be a miniversal ring of $D_{C,G_x,x}$, and let $\underline{R}_{C,G_x,x}$ denote the corresponding functor on $\mathcal{C}_\Lambda$. Since $D_{C,G}$ is prorepresentable, by versality of $R_{C,G_x,x}$ we obtain a morphism $D_{C,G} \to \underline{R}_{C,G_x,x}$ which must be an isomorphism since it induces an isomorphism on tangent spaces and both functors are prorepresented by regular complete local rings of the same dimension. Let $y \in E$ be the node lying under x. Let $D_{E,y}$ be the local deformation functor of the node $y \in E$, with miniversal family given by $R_{E,y} := \Lambda[\![T]\!] \to \Lambda[\![U,V,T]\!]/(UV-T)$. Then similarly we have $D_{E,y} \cong \underline{R}_{E,y}$. Thus, the map of universal deformation rings induced by $D_{C,G} \to D_E$ can be computed by examining the induced map on miniversal families at the nodes x and y. From the local picture of a balanced node, choosing appropriate coordinates, this map on miniversal families is given by

$$\Lambda[\![U,V,T]\!]/(UV-T) \longrightarrow \Lambda[\![u,v,t]\!]/(uv-t),$$
$$(U,V,T) \mapsto (u^e,v^e,t^e),$$

where $e=|G_x|$, and hence the map on miniversal rings is given by

$$R_{E,y}=\Lambda[\![T]\!] \longrightarrow \Lambda[\![t]\!]=R_{C,G_x,x},$$
$$T \mapsto t^e,$$

which via the isomorphisms $D_{C,G}\cong \underline{R}_{C,G_x,x}$ and $D_{E,y}\cong \underline{R}_{E,y}$ also computes the map on universal deformation rings induced by $D_{C,G}\to D_E$. Finally, the form of the miniversal families given above implies that any point of $\overline{\mathcal{H}_G}$ corresponding to a nodal curve is the specialization of a point corresponding to a smooth curve. Thus, $\mathcal{H}_G$ (resp. $\mathcal{A}dm^0(G)$) are open and dense inside $\overline{\mathcal{H}}_G$ (resp. $\mathcal{A}dm(G)$). □

COROLLARY 2.5.4. *If G cannot be generated by two elements, then $\mathcal{A}dm(G)$ is empty.*

Proof. By Proposition 2.5.3, $\mathcal{A}dm^0(G)\subset\mathcal{A}dm(G)$ is dense, so it suffices to show that if G is not 2-generated, then $\mathcal{A}dm^0(G)$ is empty. If E is an elliptic curve over an algebraically closed field k (of characteristic prime to $|G|$, since we are working universally over $\mathbb{S}=\operatorname{Spec}\mathbb{Z}[1/|G|]$), then an admissible G-cover $\pi: C\to E$ corresponds by Galois theory to a surjection $\pi_1(E_k^\circ)\to G$. Since G is prime to p, this surjection factors through the maximal prime-to-p quotient of $\pi_1(E_k^\circ)$, which is 2-generated [GR71, Exp. X, Cor. 3.10], so G must be 2-generated, as desired. □

While $\mathcal{A}dm^0(G)$ is étale over $\mathcal{M}(1)$, unlike $\mathcal{M}(G)$, the map $\mathcal{A}dm^0(G)\to\mathcal{M}(1)$ is generally not representable, hence it is generally not finite. The obstruction to representability lies in the vertical automorphism groups:

Definition 2.5.5. For a map of algebraic stacks $f:\mathcal{X}\to\mathcal{Y}$ and a T-valued point $t:T\to\mathcal{X}$, there is a homomorphism

$$f_*:\operatorname{Aut}_{\mathcal{X}(T)}(t)\to\operatorname{Aut}_{\mathcal{Y}(T)}(f(t)).$$

The *vertical automorphism group* of t (relative to f) is by definition the kernel of f_*. Thus if $\mathcal{X},\mathcal{Y}$ are Deligne-Mumford, then f is representable if and only if f_* is injective on geometric points [ACV03, Lemma 4.4.3].

For a T-valued point $t:T\to\mathcal{X}$ where $\mathcal{X}$ is a stack equipped with a map to $\overline{\mathcal{M}(1)}$ (this will essentially always be the case in this paper), its *vertical automorphism group* is by default defined to be its vertical automorphism group relative to the map to $\overline{\mathcal{M}(1)}$, and we will denote it by $\operatorname{Aut}^v(t)$.

For a geometric point $\operatorname{Spec}\Omega \to \mathcal{A}dm^0(G)$ corresponding to an admissible G-cover $\pi : C \to E$, its vertical automorphism group is the group of the G-equivariant automorphisms σ of C which induce the identity on E — i.e., which satisfy $\pi\circ\sigma = \pi$. Any G-equivariant automorphism σ of C inducing the identity on E restricts to an automorphism of the G-torsor $\pi : \pi^{-1}(E_{\text{gen}}) \to E_{\text{gen}}$. Since C is smooth and connected, it is irreducible, so $\pi^{-1}(E_{\text{gen}})$ is also irreducible, so every automorphism of $\pi^{-1}(E_{\text{gen}}) \to E_{\text{gen}}$ is given by the action of some $g \in G$. Since the automorphism is G-equivariant, we must have $g \in Z(G)$. We have proved the following:

PROPOSITION 2.5.6. *The vertical automorphism groups of geometric points of $\mathcal{A}dm^0(G)$ are isomorphic to $Z(G)$.*

Thus, if one takes $\mathcal{A}dm^0(G)$ and considers all morphisms as defined "modulo $Z(G)$," then the corresponding fibered category should be representable over $\mathcal{M}(1)$. This is achieved by the process of *rigidification* (see [ACV03, §5], [Rom05, §5]). The statement is the following:

THEOREM 2.5.7 ([ACV03, Th. 5.1.5]). *Let H be a flat finitely presented separated group scheme over $\mathbb{S}$, and let $\mathcal{X}$ be an algebraic stack over $\mathbb{S}$. Assume that for each object $\xi \in \mathcal{X}(S)$, there is an embedding*

$$i_\xi : H(S) \hookrightarrow \operatorname{Aut}_S(\xi),$$

which is compatible with pullback, in the following sense: Let $\phi : \xi \to \eta$ be a morphism in $\mathcal{X}$ lying over a morphism of schemes $f : S \to T$, and let $g \in H(T)$; we require that the following diagram commutes:

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, "i_\xi(f^*g)"'] & \eta \arrow[d, "i_\eta(g)"] \\ \xi \arrow[r, "\phi"] & \eta. \end{tikzcd}$$

Since $\mathcal{X}$ is fibered in groupoids, this implies that $i_\xi(f^*g) = \phi^* i_\eta(g)$. Then, the rigidification of $\mathcal{X}$ by H is a stack $\mathcal{X} /\!\!/ H$, equipped with a smooth surjective finitely presented morphism $\mathcal{X} \to \mathcal{X} /\!\!/ H$ which satisfies

(a) *For any object $\xi \in \mathcal{X}(S)$ with image $\eta \in (\mathcal{X} /\!\!/ H)(S)$, we have that $H(S)$ lies in the kernel of $\operatorname{Aut}_S(\xi) \to \operatorname{Aut}_S(\eta)$.*
(b) *The morphism $\mathcal{X} \to \mathcal{X} /\!\!/ H$ is a gerbe and is universal for morphisms of stacks $\mathcal{X} \to \mathcal{Y}$ satisfying* (a) *above.*
(c) *If S is the spectrum of an algebraically closed field, then in* (1), *we have $\operatorname{Aut}_S(\eta) = \operatorname{Aut}_S(\xi)/H(S)$.*
(d) *If $c : \mathcal{X} \to X$ is a coarse moduli space for $\mathcal{X}$, then by* (b) *it factorizes through a unique morphism $c' : \mathcal{X} /\!\!/ H \to X$, which is also a coarse moduli*

space for $\mathcal{X} /\!\!/ H$. In particular, $\mathcal{X} \to \mathcal{X} /\!\!/ H$ induces a homeomorphism on topological spaces.

(e) *If $\mathcal{X}$ is Deligne-Mumford, then so is $\mathcal{X} /\!\!/ H$, and the map $\mathcal{X} \to \mathcal{X} /\!\!/ H$ is étale.*

(f) *If $\mathcal{X}$ is smooth (resp. proper), then $\mathcal{X} /\!\!/ H$ is smooth (resp. proper).*

Remark 2.5.8. In [ACV03] this rigidification would be denoted $\mathcal{X}^H$, but since it seems closer to taking a quotient than taking fixed points, we will use the notation "$\mathcal{X} /\!\!/ H$" as introduced in [Rom05].

Proof. Everything but (f) and the gerbiness in (b) is [ACV03, Th. 5.1.5]. That $\mathcal{X} \to \mathcal{X} /\!\!/ H$ is a gerbe follows from the explicit description of $\mathcal{X} /\!\!/ H$ given in [ACV03, §5.1.7], the key fact being that sheafification/stackification is locally surjective on sections/objects. If $\mathcal{X}$ is smooth, then it admits a smooth covering by a smooth $\mathbb{S}$-scheme $U \to \mathcal{X}$, but then $U \to \mathcal{X} \to \mathcal{X} /\!\!/ H$ is a smooth covering as well, so $\mathcal{X} /\!\!/ H$ is also smooth. If $\mathcal{X}$ is proper, then using the fact that $\mathcal{X} \to \mathcal{X} /\!\!/ H$ is a gerbe, it is straightforward to check that $\mathcal{X} /\!\!/ H \to \mathbb{S}$ satisfies the valuative criteria of properness [Stacks, 0CLZ]. □

Applying the theorem to $H = Z(G)$, by Theorem 2.5.7(b) we obtain a canonical factorization

$$\mathcal{A}dm(G) \to \mathcal{A}dm(G) /\!\!/ Z(G) \to \overline{\mathcal{M}(1)}. \tag{2.7}$$

Definition 2.5.9. Let $\overline{\mathcal{M}(G)} := \mathcal{A}dm(G) /\!\!/ Z(G)$. As usual we will write $\overline{M(G)}$ for its coarse moduli space.

Next we record some of the basic properties of $\overline{\mathcal{M}(G)}$ and its relation to $\mathcal{A}dm(G)$.

Proposition 2.5.10. *We work over $\mathbb{S} = \operatorname{Spec} \mathbb{Z}[1/|G|]$. Let $\overline{\mathcal{M}(G)}^0 := \mathcal{A}dm^0(G) /\!\!/ Z(G)$ be the open substack classifying smooth covers.*

(a) *The map $\mathcal{A}dm(G) \to \overline{\mathcal{M}(G)}$ induces an isomorphism on coarse spaces $Adm(G) \xrightarrow{\sim} \overline{M(G)}$. In particular, it induces a homeomorphism on topological spaces. Both stacks are empty if G cannot be generated by two elements.*

(b) *Let φ be the map*

$$\varphi : \mathcal{A}dm^0(G) \to \mathcal{M}(G)$$

sending an admissible cover $\pi : C \to E$ to the G-structure on E determined by the G-torsor $C_{\text{gen}} \to E_{\text{gen}} = E^\circ := E - O$. Then φ is an étale gerbe and factors through an isomorphism

$$\overline{\mathcal{M}(G)}^0 := \mathcal{A}dm^0(G) /\!\!/ Z(G) \xrightarrow{\sim} \mathcal{M}(G).$$

In particular, we find that $\mathcal{M}(G)$ *is naturally isomorphic to an open dense substack of* $\overline{\mathcal{M}(G)}$.

(c) *The stack* $\overline{\mathcal{M}(G)}$ *is smooth and proper of relative dimension* 1 *over* $\mathbb{S}$.

(d) *The map* $\overline{\mathcal{M}(G)} \to \overline{\mathcal{M}(1)}$ *is flat, proper, and quasi-finite.*

(e) *The scheme* $\overline{M(G)}$, *and hence* $Adm(G)$, *is smooth and proper of relative dimension* 1 *over* $\mathbb{S}$.

(f) *Let* $f : G_1 \to G_2$ *be a surjection of finite* 2*-generated groups, inducing a canonical isomorphism* $\psi_f : G_2 \xrightarrow{\sim} G_1/\operatorname{Ker}(f)$. *Consider the functor*

$$f_* : \mathcal{A}dm(G_1) \to \mathcal{A}dm(G_2)$$

sending an admissible G_1*-cover* $\pi : C \to E$ *to the cover* $C/\operatorname{Ker}(f) \to E$ *equipped with an action of* G_2 *via* ψ_f. *Its behavior on morphisms is defined using the universal property of quotients. Then* f_* *induces a proper surjective and quasi-finite map*

$$\overline{\mathcal{M}(f)} : \overline{\mathcal{M}(G_1)} \to \overline{\mathcal{M}(G_2)}$$

of stacks over $\overline{\mathcal{M}(1)}$ *whose restriction to* $\mathcal{M}(G_1)$ *agrees with the map described in Theorem* 2.5.2(6).

Proof. We begin with (a). If G cannot be generated by two elements, then by Corollary 2.5.4, $\mathcal{A}dm(G)$ is empty, so $\overline{\mathcal{M}(G)}$ is empty. The rest of (a) follows from Theorem 2.5.7(d). Part (c) follows from Theorem 2.5.7(f).

For (b), we first show that the map $p : \overline{\mathcal{M}(G)}^0 \to \mathcal{M}(1)$ is finite étale. By Theorem 2.5.7(c) and Proposition 2.5.6, p is representable. Next, since $\overline{\mathcal{M}(G)}$ is proper, the map $\overline{\mathcal{M}(G)} \to \overline{\mathcal{M}(1)}$ is also proper, so $\overline{\mathcal{M}(G)}^0 \to \mathcal{M}(1)$ is also proper. By Proposition 2.5.3, $\mathcal{A}dm^0(G) \to \mathcal{M}(1)$ is étale. By Theorem 2.5.7(e), $\mathcal{A}dm^0(G) \to \mathcal{A}dm^0(G) /\!\!/ Z(G) = \overline{\mathcal{M}(G)}^0$ is étale surjective, so since $p : \overline{\mathcal{M}(G)}^0 \to \mathcal{M}(1)$ is representable, we find that p is also étale [Stacks, 0CIL]. Thus p is representable, proper, and étale, so it is finite étale [Stacks, 02LS]. Thus to show that the map $\overline{\mathcal{M}(G)}^0 \to \mathcal{M}(G)$, it would suffice to show that it induces a bijection on geometric fibers over $\mathcal{M}(1)$. This follows from the observation that if E is an elliptic curve over an algebraically closed field k, then every admissible G-cover of E restricts to give a G-torsor over E°, and conversely every G-torsor over E° (necessarily tamely ramified over $O \in E$ since we are working universally over $\mathbb{S} = \operatorname{Spec} \mathbb{Z}[1/|G|]$) extends by normalization to an admissible G-cover of E, and these processes are mutually inverse. Thus $\overline{\mathcal{M}(G)}^0 \cong \mathcal{M}(G)$ as desired. Finally, to see that $\overline{\mathcal{M}(G)}^0$, and hence $\mathcal{M}(G)$ is open dense inside $\overline{\mathcal{M}(G)}$, it suffices to check that $\mathcal{A}dm^0(G) \subset \mathcal{A}dm(G)$ is open dense, but this follows from Proposition 2.5.3. This proves (b).

For (d), flatness follows from the fact that $\mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ is flat (see Theorem 2.1.11(b)) and that $\mathcal{A}dm(G) \to \overline{\mathcal{M}(G)}$ is flat and surjective. Properness follows from the fact that it is a map of proper stacks. Quasi-finiteness follows from the same property for the map $\mathcal{A}dm(G) \to \overline{\mathcal{M}(1)}$ (see Theorem 2.1.11(b)).

Part (e) follows from Lemma 2.1.10.

For (f), first note that by Proposition 2.4.6 and Lemma 2.4.1, the quotient $C/\operatorname{Ker}(f)$ exists. Next we check that $C/\operatorname{Ker}(f) \to E$ satisfies conditions (1)–(6) of an admissible $G_1/\operatorname{Ker}(f) \cong G_2$-cover (Definition 2.1.4). Part (1) follows from Proposition 2.4.4. Part (6) is immediate from the same property of C. Part (3) is Galois theory. Parts (2), (4), and (5) are local questions and follow from the explicit étale local description of admissible G-covers.

By Lemma 2.4.1(c), formation of the quotient $C/\operatorname{Ker}(f)$ commutes with arbitrary base change so f_* is a morphism of stacks. By the universal property of rigidification (cf. Theorem 2.5.7(b)), the composition

$$\mathcal{A}dm(G_1) \longrightarrow \mathcal{A}dm(G_2) \longrightarrow \mathcal{A}dm(G_2) /\!\!/ Z(G_2) = \overline{\mathcal{M}(G_2)}$$

factors uniquely via

$$\mathcal{A}dm(G_1) \longrightarrow \mathcal{A}dm(G_1) /\!\!/ Z(G_1) = \overline{\mathcal{M}(G_1)} \longrightarrow \overline{\mathcal{M}(G_2)},$$

and $\overline{\mathcal{M}(f)}$ will be defined to be the second arrow in the above factorization. As a map between proper stacks, it is proper. As a map between Deligne-Mumford stacks quasi-finite over $\overline{\mathcal{M}(1)}$, it is quasi-finite. By construction its restriction to the smooth locus agrees with the map described in Theorem 2.5.2(6). Since the induced maps over $\mathcal{M}(1)$ are surjective and $\mathcal{M}(G) \subset \overline{\mathcal{M}(G)}$ is open dense, the surjectivity of $\overline{\mathcal{M}(f)} : \overline{\mathcal{M}(G_1)} \to \overline{\mathcal{M}(G_2)}$ is a consequence of its properness. $\square$

Definition 2.5.11. In light of Proposition 2.5.10, we see that $\overline{\mathcal{M}(G)}$ is a smooth modular compactification of the moduli stack of elliptic curves with G-structures. If k is a field, then a k-point of $\overline{\mathcal{M}(G)}$ (resp. $\overline{M(G)}, \mathcal{A}dm(G)$, $Adm(G)$) not lying in $\mathcal{M}(G)$ (resp. $M(G), \mathcal{A}dm^0(G), Adm^0(G)$) is called a *cusp* or a *cuspidal object*.

Remark 2.5.12. The terminology of "cusp" comes from the fact that the schemes $M(G)_{\mathbb{C}}$ are disjoint unions of quotients of the upper half plane by finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ acting via Möbius transformations [Che18, §3.3.5]. Each component of $M(G)_{\mathbb{C}}$ can be viewed as a modular curve for a possibly non-congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, whose cusps correspond exactly to the points in $\overline{M(G)}_{\mathbb{C}} - M(G)_{\mathbb{C}}$.

2.5.3. *A Galois theoretic mantra*. We will often need to use the Galois correspondence to translate between the category of stacks finite étale over $\mathcal{M}(1)_{\overline{\mathbb{Q}}}$ and finite sets with $\pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}})$-action. Some setup is required to make this precise, so that Theorem 2.5.2 can be applied. To avoid describing the same setup repeatedly, we will gather it into the following "situation" (Situation 2.5.14 below) which we will refer to when needed. We begin with a definition of "equivalence classes of G-structures."

For any 2-generated group G, let $\mathcal{T}_G : \mathcal{M}(1) \to \underline{\textbf{Sets}}$ be the sheaf of G-structures defined in Section 2.5. By functoriality of $\mathcal{T}_G$ in G (cf. Theorem 2.5.2(6)), we obtain an action of $\mathrm{Aut}(G)$ on $\mathcal{T}_G$. By looking at geometric fibers, we find that $\mathrm{Inn}(G)$ acts trivially and that the induced action of $\mathrm{Out}(G)$ on $\mathcal{T}_G$ is *free*. In particular, we get an action of $\mathrm{Out}(G)$ on the stack $\mathcal{M}(G)$ over $\mathcal{M}(1)$. Thus for any subgroup $A \subset \mathrm{Out}(G)$, the quotient $\mathcal{T}_G/A$ is also a finite locally constant sheaf on $\mathcal{M}(G)$.

Definition 2.5.13. Let $A \subset \mathrm{Out}(G)$ be a subgroup. The sections of $\mathcal{T}_G/A$ over an elliptic curve E/S will be called $(G|A)$-structures on E/S. The moduli stack of elliptic curves with $(G|A)$-structures is the quotient $\mathcal{M}(G)/A$, defined as the stack over $\mathcal{M}(1)$ associated to the quotient sheaf $\mathcal{T}_G/A$. When $A = \mathrm{Out}(G)$, we will call them *absolute G-structures*,[22] and the corresponding stack $\mathcal{M}(G)/\mathrm{Out}(G)$ will be denoted $\mathcal{M}(G)^{\mathrm{abs}}$.

Situation 2.5.14. Let E be an elliptic curve over $\overline{\mathbb{Q}}$. Let $t \in E^\circ(\mathbb{C})$ be a point. Let $\Pi := \pi_1^{\mathrm{top}}(E^\circ(\mathbb{C}), t)$. Let a, b be a basis for Π with intersection number $+1$. (We will call this a "positively oriented basis.") Let $x_E : \mathrm{Spec}\,\overline{\mathbb{Q}} \to \mathcal{M}(1)$ be the geometric point corresponding to E. Recall that $\mathrm{Aut}^+(\Pi)$ is defined to be the subgroup of $\mathrm{Aut}(\Pi)$ of elements which induce determinant 1 automorphisms of $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$, and $\mathrm{Out}^+(\Pi) := \mathrm{Aut}^+(\Pi)/\mathrm{Inn}(\Pi)$. By Theorem 2.5.2(5), there is a canonical map $i : \mathrm{Out}^+(\Pi) \hookrightarrow \pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, x_E)$ which is injective with dense image. It induces an isomorphism $\widehat{\mathrm{Out}^+(\Pi)} \cong \pi_1(\mathcal{M}(1)_{\overline{\mathbb{Q}}}, x_E)$, which we will use to identify the two groups. In particular, we obtain an action of $\mathrm{Out}^+(\Pi)$ (and hence $\mathrm{Aut}^+(\Pi)$) on the geometric fiber over x_E of any stack finite étale over $\mathcal{M}(1)$. Let G be a finite 2-generated group, and let

$$\mathfrak{f} : \mathcal{M}(G) \to \mathcal{M}(1)$$

be the forgetful map. The fiber $\mathfrak{f}^{-1}(x_E)$ is the set of isomorphism classes of geometrically connected G-torsors over E°. Taking monodromy representations (see Section 2.3) gives a bijection

$$\alpha_{G,x_E} : \mathfrak{f}^{-1}(x_E) \xrightarrow{\sim} \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$$

[22]We distinguish this special case because absolute structures have a natural moduli interpretation. Specifically, let $k = \overline{k}$ have characteristic coprime to $|G|$, and E/k an elliptic curve. An absolute G-structure on E is represented by a G-torsor over E°. Two absolute G-structures over $k = \overline{k}$ are the same if their corresponding torsors are isomorphic *as covers of* E°. Namely, we do not require the isomorphism to be G-equivariant. Thus an absolute G-structure on E amounts to giving a finite Galois cover of E, branched only above the origin, whose Galois group is isomorphic to G (cf. [BBCL22, §2.4])

which is $\mathrm{Out}^+(\Pi)$-equivariant, where $\mathrm{Out}^+(\Pi)$ acts on $\mathfrak{f}^{-1}(x_E)$ via i (cf. Theorem 2.5.2). Via this bijection, the Higman invariant can also be expressed at the level of $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$ (cf. Remark 2.3.3).

To be precise, there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{f}^{-1}(x_E) \arrow[r, "\alpha_{G,x_E}"] \arrow[dr, "\mathrm{Hig}"'] & \mathrm{Epi}^{\mathrm{ext}}(\Pi, G) \arrow[d, "\varphi\mapsto\varphi({[b,a]})"] \\
& \mathrm{Cl}(G),
\end{tikzcd}$$

where $\mathrm{Cl}(G)$ denotes the set of conjugacy classes of G. Let $\mathcal{C}$ be the category whose objects are 2-generated finite groups, and where morphisms are surjections. Let $\underline{\mathbf{FinSets}}_{\mathrm{Out}^+(\Pi)}$ be the category of finite sets with $\mathrm{Out}^+(\Pi)$-action, and let $\mathrm{FEt}_{\mathcal{M}(1)_{\overline{\mathbb{Q}}}}$ be the category of finite étale maps to $\mathcal{M}(1)$. Let $\mathcal{M} : \mathcal{C} \to \mathrm{FEt}_{\mathcal{M}(1)_{\overline{\mathbb{Q}}}}$ be the epimorphism-preserving functor of Theorem 2.5.2(6). Let $F_{x_E} : \mathrm{FEt}_{\mathcal{M}(1)_{\overline{\mathbb{Q}}}} \to \underline{\mathbf{FinSets}}_{\mathrm{Out}^+(\Pi)}$ be the fiber functor at x_E. (This is an equivalence of categories by Galois theory.) Then the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{M}"] \arrow[dr, "\mathrm{Epi}^{\mathrm{ext}}(\Pi,-)"'] & \mathrm{FEt}_{\mathcal{M}(1)_{\overline{\mathbb{Q}}}} \arrow[d, "F_{x_E}"] \\
& \underline{\mathbf{FinSets}}_{\mathrm{Out}^+(\Pi)}
\end{tikzcd}$$

2-commutes, in the sense that the two paths are isomorphic as functors, with the isomorphism defined using the isomorphisms α_{G,x_E}. In particular, if $A \subset \mathrm{Out}(G)$ is a subgroup, if $g : \mathcal{M}(G) \to \mathcal{M}(G)/A$ is the quotient map, and if $\mathfrak{f}_A : \mathcal{M}(G)/A \to \mathcal{M}(1)$ is the forgetful map, then the commutative diagram

$$\begin{tikzcd}
\mathcal{M}(G) \arrow[r, "g"] \arrow[dr, "\mathfrak{f}"'] & \mathcal{M}(G)/A \arrow[d, "\mathfrak{f}_A"] \\
& \mathcal{M}(1)
\end{tikzcd}$$

induces the diagram

$$\begin{tikzcd}
\mathfrak{f}^{-1}(x_E) \arrow[r, "\cong"] \arrow[d, "g"] & \mathrm{Epi}^{\mathrm{ext}}(\Pi, G) \arrow[d, "h"] \\
\mathfrak{f}_A^{-1}(x_E) \arrow[r, "\cong"] & \mathrm{Epi}^{\mathrm{ext}}(\Pi, G)/A,
\end{tikzcd}$$

where h is the quotient map, and the bottom bijection is also $\mathrm{Out}^+(\Pi)$-equivariant.

3. Degrees of components of $Adm(G)$ over $\overline{M(1)}$

In this section, after recalling the definitions and formalism of the cotangent sheaf, relative dualizing sheaf, ramification divisor, and degrees of line bundles on 1-dimensional stacks, in Section 3.5 we establish the basic form of our main congruence on the degrees of components of $Adm(G)$ over $\overline{M(1)}$.

3.1. *The cotangent and relative dualizing sheaves for universal families over moduli stacks.*

Definition 3.1.1. We say that a representable map $\mathcal{C} \to \mathcal{X}$ of algebraic stacks is a *prestable curve* if for any map $S \to \mathcal{X}$ with S a scheme, the pullback $\mathcal{C}_S \to S$ is a prestable curve in the sense of Definition 2.1.2.

The purpose of this section is to review the canonical map $\Omega_{\mathcal{C}/\mathcal{X}} \to \omega_{\mathcal{C}/\mathcal{X}}$ when $\mathcal{C} \to \mathcal{X}$ is a representable map of algebraic stacks which is a prestable curve. We work universally over a base scheme $\mathbb{S}$.

3.1.1. *The sheaf of relative differentials*. Let $\mathbb{S}$ be a scheme. Let $\mathcal{C}$ be an algebraic stack over $\mathbb{S}$. We can view $\mathcal{C}$ as a site by giving $\mathcal{C}$ the inherited topology from $(\underline{\mathbf{Sch}}/\mathbb{S})_{fppf}$. Recall that this means that given an object $\xi : T \to \mathcal{C}$ with T a scheme, a covering of ξ is given by a family $\{\xi_i\}$ of objects of $\mathcal{C}$ such that each ξ_i is given by $\xi_i : T_i \xrightarrow{t_i} T \to \mathcal{C}$ such that $\{t_i : T_i \to T\}$ is a covering family in $(\underline{\mathbf{Sch}}/\mathbb{S})_{fppf}$. A sheaf on $\mathcal{C}$ is just a sheaf on the corresponding site [Stacks, 06TF]. The structure sheaf on $\mathcal{C}$ is the sheaf of rings $\mathcal{O}_{\mathcal{C}}$ given by $(U \to \mathcal{C}) \mapsto \Gamma(U, \mathcal{O}_U)$. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{C}}$-modules is quasicoherent if its restriction to each $T \to \mathcal{C}$ is a quasicoherent sheaf on T.

Let $f : \mathcal{C} \to \mathcal{X}$ be a representable map of algebraic stacks. Given $t : T \to \mathcal{C}$ with T a scheme, consider the diagram

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "\tilde{t}"] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, "t"] \arrow[u, bend left, "\tau"] & \mathcal{X}, \end{tikzcd} \tag{3.1}$$

where the outer square is cartesian, and τ is the section induced by t. This diagram is commutative if one ignores τ. Since $\mathcal{C}_T$ is a scheme, it makes sense to define $\Omega_{\mathcal{C}/\mathcal{X}}$ to be the presheaf on $\mathcal{C}$ given by

$$\Omega_{\mathcal{C}/\mathcal{X}}(t) := \Gamma(T, \tau^*\Omega_{\mathcal{C}_T/T})$$

We note that given a diagram of the form (3.1), it "factors through" a diagram (commutative if one ignores sections)

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "\tilde{\tau}"] \arrow[d] & \mathcal{C}_{\mathcal{C}_T} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ T \arrow[r, "\tau"] & \mathcal{C}_T \arrow[r, "f\circ\tilde{t}"'] \arrow[ur, "\tilde{t}"] \arrow[u, bend left, "\tilde{\tau}"] & \mathcal{X}, \end{tikzcd}$$

where the right square is cartesian, and $\tilde{\tau}$ is the section induced by $\tilde{t}$; this implies that the outer rectangle is cartesian, hence the left square is cartesian. Since sheaves of relative differentials commute with base change, we have

$$\Omega_{\mathcal{C}/\mathcal{X}}(\tilde{t}:\mathcal{C}_T \to \mathcal{C}) := \Gamma(\mathcal{C}_T, \tilde{\tau}^*\Omega_{\mathcal{C}_{\mathcal{C}_T}/\mathcal{C}_T}) \cong \Gamma(\mathcal{C}_T, \Omega_{\mathcal{C}_T/T}). \tag{3.2}$$

The restriction maps of $\Omega_{\mathcal{C}/\mathcal{X}}$ are defined as follows. Given a map $s : S \to \mathcal{C}$ and a map $p : S \to T$ with $t \circ p = s$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}_S \arrow[r, "\tilde{p}"] \arrow[d] & \mathcal{C}_T \arrow[r, "\tilde{t}"] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ S \arrow[u, bend left, "\sigma"] \arrow[r, "p"] & T \arrow[u, bend left, "\tau"] \arrow[ur, "t"] \arrow[r] & \mathcal{X} \end{tikzcd}$$

with both squares cartesian and where the section σ is induced by $s = t \circ p$. Note that $\mathcal{C}_S, \mathcal{C}_T$ are schemes since f is representable. From the diagram we obtain natural restriction maps on global sections

(3.3)
$$\rho_{p,t,s} : \Omega_{\mathcal{C}/\mathcal{X}}(t) := \Gamma(T, \tau^*\Omega_{\mathcal{C}_T/T}) \longrightarrow \Gamma(S, p^*\tau^*\Omega_{\mathcal{C}_T/T}) \xrightarrow{\sim} \Gamma(S, \sigma^*\tilde{p}^*\Omega_{\mathcal{C}_T/T})$$
$$\xrightarrow{\sim} \Gamma(S, \sigma^*\Omega_{\mathcal{C}_S/S}) =: \Omega_{\mathcal{C}/\mathcal{X}}(s).$$

The first map is the usual pullback map on sections. The second map comes from the unique isomorphism $p^*\tau^* \cong \sigma^*\tilde{p}^*$ of functors $\underline{\mathbf{Mod}}(\mathcal{O}_{\mathcal{C}_T}) \to \underline{\mathbf{Mod}}(\mathcal{O}_S)$, the uniqueness coming from the fact that $p^*\tau^*$ and $\sigma^*\tilde{p}^*$ are both left adjoints of $\tau_* \circ p_* = \tilde{p}_* \circ \sigma_* = (\tau \circ p)_*$. The final map is induced by the unique isomorphism $\tilde{p}^*\Omega_{\mathcal{C}_T/T} \cong \Omega_{\mathcal{C}_S/S}$ coming from the universal property of the sheaf of differentials.

The restriction of $\Omega_{\mathcal{C}/\mathcal{X}}$ to any scheme $t : T \to \mathcal{C}$ is just the quasicoherent sheaf on $(\underline{\mathbf{Sch}}/T)_{fppf}$ associated to the usual quasicoherent sheaf $\tau^*\Omega_{\mathcal{C}_T/T}$ on T. Thus $\Omega_{\mathcal{C}/\mathcal{X}}$ is a quasicoherent sheaf, and by (3.2) it agrees with the usual sheaf of relative differentials when $\mathcal{C}, \mathcal{X}$ is are schemes.

Definition 3.1.2. Let $f : \mathcal{C} \to \mathcal{X}$ be a representable morphism of algebraic stacks. Let $\Omega_{\mathcal{C}/\mathcal{X}}$ denote the sheaf of relative differentials, as defined above.

Alternatively, the sheaf $\Omega_{\mathcal{C}/\mathcal{X}}$ can be defined on a presentation for $\mathcal{C}/\mathcal{X}$. Namely, let U be a scheme, and let $U \to \mathcal{X}$ now be a smooth surjective morphism. Then $\mathcal{C}_U := \mathcal{C} \times_{\mathcal{X}} U \to \mathcal{C}$ is also smooth and surjective. Let $R := \mathcal{C}_U \times_{\mathcal{C}} \mathcal{C}_U$ and $S := U \times_{\mathcal{X}} U$. This determines a commutative diagram

$$\begin{tikzcd} R \arrow[r, shift left, "\mathrm{pr}_0"] \arrow[r, shift right, "\mathrm{pr}_1"'] \arrow[d] & \mathcal{C}_U \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r, shift left, "\mathrm{pr}_0"] \arrow[r, shift right, "\mathrm{pr}_1"'] & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

which induces isomorphisms $[\mathcal{C}_U/R] \cong \mathcal{C}$ and $[U/S] \cong \mathcal{X}$ [Stacks, 04T4]. To give a quasicoherent sheaf on $\mathcal{C}$ is the same as giving a quasicoherent sheaf $\mathcal{F}$ on $\mathcal{C}_U$ together with an isomorphism

$$\alpha : \mathrm{pr}_0^* \mathcal{F} \xrightarrow{\sim} \mathrm{pr}_1^* \mathcal{F}$$

over R satisfying a certain cocycle condition [Stacks, 0441,06WT]. Letting $\mathcal{F} = \Omega_{\mathcal{C}_U/U}$ and α the canonical isomorphism $\mathrm{pr}_0^* \Omega_{\mathcal{C}_U/U} \cong \Omega_{R/S} \cong \mathrm{pr}_1^* \Omega_{\mathcal{C}_U/U}$, one checks that α satisfies the cocycle condition, and hence this determines a sheaf $\Omega_{\mathcal{C}/\mathcal{X}}$ on $\mathcal{C}$ which agrees with our earlier definition.

3.1.2. *The relative dualizing sheaf and the canonical map $\Omega_{\mathcal{C}/\mathcal{X}} \to \omega_{\mathcal{C}/\mathcal{X}}$.* Let $f : C \to X$ be a flat proper finitely presented morphism of schemes. Recall that if X is quasi-compact quasi-separated, then the functor $Rf_* : D(\mathcal{O}_C) \to D(\mathcal{O}_X)$ has a right adjoint $f^! : D(\mathcal{O}_X) \to D(\mathcal{O}_C)$ [Stacks, 0B6S]. In this case the relative dualizing complex is $\omega^\bullet_{C/X} := f^!(\mathcal{O}_X)$, and it is equipped with a "trace map" $\mathrm{tr}_f : Rf_*\omega^\bullet_{C/X} \to \mathcal{O}_X$ coming from adjunction. For general X, the uniqueness of the adjoint implies that the relative dualizing complexes locally defined on affine opens of X glue to yield a pair $(\omega^\bullet_{C/X}, \mathrm{tr}_f)$, where $\omega^\bullet_{C/X} \in D(\mathcal{O}_X)$ and $\mathrm{tr}_f : Rf_*\omega^\bullet_{C/X} \to \mathcal{O}_X$ is such that the pair $(\omega^\bullet_{C/X}, \mathrm{tr}_f)$ restricts to the relative dualizing complex over affine opens of X [Stacks, 0E61]. Moreover, such a pair is unique up to unique isomorphism and commutes with arbitrary base change [Stacks, 0E5Z,0E60].

If, moreover, f has Cohen-Macaulay and geometrically connected fibers of constant relative dimension d, then the dualizing complex $\omega^\bullet_{C/X}$ has a unique non-zero cohomology sheaf which is in degree $-d$ [Stacks, 0BV8].[23] Let $\omega_{C/X} := H^{-d}(\omega^\bullet_{C/X})$ denote this unique non-zero cohomology sheaf, which is called the (relative) dualizing sheaf for f.

Definition 3.1.3. Let $f : \mathcal{C} \to \mathcal{X}$ be a representable flat proper finitely presented morphism of algebraic stacks with geometrically connected and Cohen-Macaulay fibers. The relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{X}}$ is defined as follows. Given any diagram of the form (3.1), define

$$\omega_{\mathcal{C}/\mathcal{X}}(t) := \Gamma(T, \tau^*\omega_{\mathcal{C}_T/T}).$$

As in the case for $\Omega_{\mathcal{C}/\mathcal{X}}$, we obtain natural restriction maps $\rho_{p,t,s}$ as in (3.3). Since relative dualizing sheaves commute with arbitrary base change, the same

[23]Technically, in order to use this, one should first reduce to the case where X is affine, and then use Noetherian approximation to note that f is the base change of a morphism $f_0 : C_0 \to X_0$ where X_0 is of finite type over $\mathbb{Z}$, and f_0 is also flat proper finitely presented with Cohen-Macaulay fibers [Stacks, 01ZA,081C,045U]. In this situation [Stacks, 0BV8] applies, and by pullback we deduce the result for f.

discussion as above implies that the restriction maps are compatible and define a quasicoherent sheaf on $\mathcal{C}$, which is called the *relative dualizing sheaf* of $\mathcal{C}/\mathcal{X}$.

PROPOSITION 3.1.4. *Let $f : \mathcal{C} \to \mathcal{X}$ be a representable map of algebraic stacks which is a prestable curve. Then there is a canonical map*

$$\psi : \Omega_{\mathcal{C}/\mathcal{X}} \longrightarrow \omega_{\mathcal{C}/\mathcal{X}}$$

which is an isomorphism where f is smooth.

Proof. Let $T \to \mathcal{X}$ be a map with T a scheme. The easiest way to define ψ is to note that $\Omega_{\mathcal{C}_T/T}$ viewed as an object of the derived category $\mathcal{D}(\mathcal{O}_{\mathcal{C}_T})$ is a perfect complex [Knu83, Cor. 3.3], and hence we may construct $\omega_{\mathcal{C}_T/T}$ as the determinant $\det(\Omega_{\mathcal{C}_T/T})$ in the sense of [KM76].[24] In this case we have a canonical map $\psi_T = \psi_{T\to\mathcal{X}} : \Omega_{\mathcal{C}_T/T} \to \det(\Omega_{\mathcal{C}_T/T}) = \omega_{\mathcal{C}_T/T}$ as described in [Knu83, §1], which is an isomorphism on the smooth locus. It is clear from the description of ψ_T that they commute with base change relative to any map $T' \to T$. This implies that the family of maps $\{\psi_T\}$ define a morphism $\psi : \Omega_{\mathcal{C}/\mathcal{X}} \to \omega_{\mathcal{C}/\mathcal{X}}$ as desired.[25] $\square$

3.2. *The universal family over $\mathcal{A}dm(G)$ and its reduced ramification divisor*. Let G be a finite group. We work universally over a $\mathbb{Z}[1/|G|]$-scheme $\mathbb{S}$. The universal admissible G-cover

$$\mathcal{C}(G) \longrightarrow \mathcal{E}(G) \longrightarrow \mathcal{A}dm(G) \tag{3.4}$$

is defined as follows. The objects of $\mathcal{E}(G)$ over a scheme T are pairs $(\pi : C \to E, \sigma)$, where π is an admissible G-cover of a 1-generalized elliptic curve E over T and $\sigma : T \to E$ is a section. Morphisms are morphisms in the category $\mathcal{A}dm(G)$ respecting the sections σ. The map $\mathcal{E}(G) \to \mathcal{A}dm(G)$ is given by forgetting σ. Similarly, the objects of $\mathcal{C}(G)$ over a scheme T are pairs $(\pi : C \to E, \sigma)$ where π is again an admissible G-cover of a 1-generalized elliptic curve E over T, and $\sigma : T \to C$ is a section, with morphisms similarly defined. The map $\mathcal{C}(G) \to \mathcal{E}(G)$ sends $(\pi : C \to E, \sigma)$ to $(\pi : C \to E, \pi \circ \sigma)$. In particular, we find that for a geometric point $\overline{z}$ of $\mathcal{C}(G)$ mapping to $\overline{x}$ in $\mathcal{A}dm(G)$, the automorphism group of $\overline{z}$ is precisely the group of automorphisms of the admissible G-cover $\pi_{\overline{x}} : \mathcal{C}(G)_{\overline{x}} \to \mathcal{E}(G)_{\overline{x}}$ which fix $\overline{z}$.

It follows from the above discussion that the maps (3.4) are representable and for any scheme T and map $T \to \mathcal{A}dm(G)$ given by an admissible G-cover

[24] By [Kle80, Cor. 19] this sheaf is dualizing and hence agrees with our definition of $\omega_{\mathcal{C}_T/T}$ above. Also see [BR11, §4.1.1-4.1.2] and [Liu02, §6.4].

[25] Alternatively, using the more traditional definition of $\omega_{\mathcal{C}_T/T}$, the map ψ_T is also described in [Stacks, 0E9Z], though it is less clear from this description that it commutes with base change.

$\pi : C \to E$ over T, the pullback of (3.4) to T is canonically isomorphic to $C \xrightarrow{\pi} E \to T$.

Let $\mathcal{R}_{\mathcal{C}(G)/\mathcal{E}(G)}$ be the strictly full subcategory of $\mathcal{C}(G)$ whose objects over a scheme T are pairs $(\pi : C \to E, \sigma)$ where $\sigma : T \to C$ factors (uniquely) through the closed immersion $\mathcal{R}_\pi \hookrightarrow C$. If $T \to \mathcal{A}dm(G)$ is a map given by an admissible G-cover $\pi : C \to E$, then $\mathcal{R}_{\mathcal{C}(G)/\mathcal{E}(G)} \times_{\mathcal{A}dm(G)} T = \mathcal{R}_\pi$. It follows that the inclusion map $\mathcal{R}_{\mathcal{C}(G)/\mathcal{E}(G)} \subset \mathcal{C}(G)$ is a closed immersion.

Let $\mathcal{X} \subset \mathcal{A}dm(G)$ be a connected component, and let

$$\mathcal{C} \xrightarrow{\pi} \mathcal{E} \longrightarrow \mathcal{X}$$

denote the restriction of the universal family to $\mathcal{X}$. Let $\mathcal{R}_\pi = \mathcal{R}_{\mathcal{C}/\mathcal{E}}$ denote the restriction of $\mathcal{R}_{\mathcal{C}(G)/\mathcal{E}(G)}$ to $\mathcal{C}$.

Definition 3.2.1. The *reduced ramification divisor* of $\mathcal{C}/\mathcal{E}$ is the closed substack $\mathcal{R}_{\mathcal{C}/\mathcal{E}} \subset \mathcal{C}$ defined above.

Proposition 3.2.2. *The components of the reduced ramification divisor can be controlled as follows.*

(a) *Working over* $\mathbb{S} = \operatorname{Spec}\mathbb{Z}[1/|G|]$, *let* $\mathcal{X} \subset \mathcal{A}dm(G) = \mathcal{A}dm(G)_\mathbb{S}$ *be a connected component classifying covers with ramification index* e. *Let* $\mathcal{C} \xrightarrow{\pi} \mathcal{E} \to \mathcal{X}$ *be the universal family. Then the reduced ramification divisor* $\mathcal{R}_\pi = \mathcal{R}_{\mathcal{C}/\mathcal{E}}$ *is finite étale over* $\mathcal{X}$ *of degree* $|G|/e$. *Let* $\overline{z}$ *be a geometric point of* $\mathcal{R}_\pi$ *with stabilizer* $G_{\overline{z}} \le G$. *The connected components of* $\mathcal{R}_\pi$ *are all isomorphic, each Galois over* $\mathcal{X}$ *with Galois group isomorphic to a subgroup of* $N_G(G_{\overline{z}})/G_{\overline{z}}$.

(b) *Working over* $\mathbb{S} = \operatorname{Spec}\mathbb{Z}[1/|G|, \zeta_e]$, *let* $\mathcal{X} \subset \mathcal{A}dm(G) = \mathcal{A}dm(G)_\mathbb{S}$ *be a connected component classifying covers with ramification index* e. *Let* $\mathcal{C} \xrightarrow{\pi} \mathcal{E} \to \mathcal{X}$ *be the universal family. Then the conclusions of* (a) *hold and the Galois groups of components of* $\mathcal{R}_\pi$ *are moreover isomorphic to a subgroup of* $C_G(G_{\overline{z}})/G_{\overline{z}}$.

Proof. For part (a), from the discussion above, we may argue exactly as in Proposition 2.2.5(a). In part (b), the conclusions of (a) hold by base change, so it remains to justify the claim about the Galois groups. If $e = 1$, then the statement is trivial, so we may assume $e \ge 2$, and hence $\mathcal{C} \to \mathcal{X}$ has fibers of genus $g \ge 2$. Let $\mathcal{R} \subset \mathcal{R}_\pi$ be a connected component. Suppose there exists a map $f : U \to \mathcal{X}$ with U a regular integral scheme such that $f^*\mathcal{R}$ is connected. Then we may apply Proposition 2.2.5(b) to the pullback $f^*\mathcal{C} \to f^*\mathcal{E}$, which would give us the desired result. To construct the map f, let $K := \mathbb{Q}(\zeta_e)$, let $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,n}$) denote the moduli stack of smooth curves of genus g (resp. with n distinct marked points) over K (see [Knu83]), and let $\mathcal{Y}_K \subset \mathcal{X}_K$ be the open substack consisting of smooth objects. There is a natural map

$$h : \mathcal{Y}_K \to \mathcal{M}_g$$

sending an admissible cover $C \to E$ to the genus g curve C. The Riemann-Hurwitz formula together with Hurwitz's automorphism theorem implies that for fixed g, there is a large enough n such that curves of genus g (in characteristic 0) do not have any automorphisms with n fixed points, so the stack $\mathcal{M}_{g,n}$ is a scheme. Let us fix such an n. Forgetting marked points yields natural maps

$$\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n-1} \longrightarrow \cdots \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0} = \mathcal{M}_g,$$

where the source of each map is the universal family over the target. Thus the composition $\mathcal{M}_{g,n} \to \mathcal{M}_g$ is representable, smooth, and proper with geometrically connected fibers. Let $U := h^*\mathcal{M}_{g,n}$. Then since automorphisms of objects in $\mathcal{A}dm(G)$ are determined by how it behaves on the covering curve C, the map h is representable and hence U is a scheme. Since $\mathcal{Y}_K$ is regular and connected and $U \stackrel{\text{pr}}{\to} \mathcal{Y}_K$ is proper and smooth (hence open and closed) with connected fibers, U is also regular and connected, so it is a regular integral scheme. Let f be the composition

$$f : U \xrightarrow{\text{pr}} \mathcal{Y}_K \xrightarrow{i} \mathcal{X}.$$

By Proposition 2.5.10(e), the coarse scheme X of $\mathcal{X}$ is smooth over $\mathbb{S}$, hence normal, hence irreducible since it is connected, so the same is true of the coarse scheme of $\mathcal{R}$. This implies that $\mathcal{X}$ and $\mathcal{R}$ are irreducible, so the restriction $i^*\mathcal{R}$ is also irreducible. Next, the map $f^*\mathcal{R} \to i^*\mathcal{R}$ is smooth proper with geometrically connected fibers since the same is true of pr. Since $i^*\mathcal{R}$ is connected, this implies that $f^*\mathcal{R}$ is also connected, as desired. $\square$

3.3. *Restriction of the relative dualizing sheaf to a ramified section.* Let G be a finite group. We work universally over a $\mathbb{Z}[1/|G|]$-scheme $\mathbb{S}$.

PROPOSITION 3.3.1. *Let $\mathcal{X} \subset \mathcal{A}dm(G)$ be a connected component with universal family $\mathcal{C} \stackrel{\pi}{\to} \mathcal{E} \to \mathcal{X}$. Let $\sigma_O : \mathcal{X} \to \mathcal{E}$ denote the zero section. Suppose $\mathcal{C} \to \mathcal{E}$ has ramification index e above σ_O. Suppose further that we have a section $\sigma : \mathcal{X} \to \mathcal{C}$ making the following diagram commute*:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] & \mathcal{E}. \\ & \mathcal{X} \arrow[ul, "\sigma"] \arrow[u, "\sigma_O"'] \end{tikzcd}$$

Then, there is a canonical isomorphism $\varphi : \sigma_O^*\omega_{\mathcal{E}/\mathcal{X}} \cong (\sigma^*\omega_{\mathcal{C}/\mathcal{X}})^{\otimes e}$.

Proof. Let $T \to \mathcal{X}$ be a map with T a scheme, corresponding to an admissible G-cover $C \stackrel{\pi}{\to} E \to T$. Abusing notation, let σ, σ_O also denote the sections of C, E pulled back from σ, σ_O, with respective sheaves of ideals $\mathcal{J} \subset \mathcal{O}_C, \mathcal{I} \subset \mathcal{O}_E$. Since $\pi \circ \sigma = \sigma_O$, π induces a map $\pi^*\mathcal{I} \to \mathcal{J}$. The étale local picture of an admissible cover above a marking implies that this map factors through an isomorphism $\pi^*\mathcal{I} \xrightarrow{\sim} \mathcal{J}^e \subset \mathcal{J}$. On the other hand, since σ, σ_O land in the smooth loci of C and E, the conormal exact sequence induces canonical isomorphisms

$$\mathcal{J}/\mathcal{J}^2 \cong \sigma^*\Omega_{C/T}, \qquad \mathcal{I}/\mathcal{I}^2 \cong \sigma_O^*\Omega_{E/T}.$$

Thus we obtain canonical isomorphisms

$$\begin{aligned}\sigma_O^*\Omega_{E/T} &\cong \mathcal{I}/\mathcal{I}^2 \cong \mathcal{I}\otimes_{\mathcal{O}_E}\mathcal{O}_E/\mathcal{I} \cong \sigma_O^*\mathcal{I}\\ &\cong \sigma^*\pi^*\mathcal{I} \cong \sigma^*\mathcal{J}^e \cong (\mathcal{J}/\mathcal{J}^2)^{\otimes e} \cong (\sigma^*\Omega_{C/T})^{\otimes e}\end{aligned} \tag{3.5}$$

compatible with any base change. Since the canonical map $\psi_T : \Omega_{C/T} \to \omega_{C/T}$ of Proposition 3.1.4 is an isomorphism on the smooth locus and is compatible with base change (and similarly for E/T), (3.5) induces an isomorphism φ_T : $\sigma_O^*\omega_{E/T} \cong (\sigma^*\omega_{C/T})^{\otimes e}$ also compatible with base change. This compatibility implies that the family $\{\varphi_T\}$ defines an isomorphism $\varphi : \sigma_O^*\omega_{\mathcal{E}/\mathcal{X}} \cong (\sigma^*\omega_{\mathcal{C}/\mathcal{X}})^{\otimes e}$ as desired. $\square$

3.4. *Degree formalism for line bundles on* 1*-dimensional stacks*. The main congruence described in the introduction (Theorem 1.1.1) originates from a congruence on the degree of a certain line bundle on $\mathcal{A}dm(G)$. Here we recall the notion of degree for a line bundle on a proper 1-dimensional algebraic stack and some of its basic properties.

Given a finite flat morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, given a map $y : \operatorname{Spec} k \to \mathcal{Y}$ with k a field, its degree $\deg_y(f)$ at y is the k-rank of the fiber $\mathcal{X} \times_{\mathcal{Y},y} \operatorname{Spec} k$. This integer is locally constant on $\mathcal{Y}$. If $\mathcal{Y}$ is connected, we denote it simply by $\deg(f) \in \mathbb{Z}$. For a line bundle $\mathcal{L}$ on a proper scheme X of pure dimension 1 over a field k, its degree is

$$\deg(\mathcal{L}) = \deg_k(\mathcal{L}) := \chi_k(X,\mathcal{L}) - \chi_k(X,\mathcal{O}_X) \qquad \text{[Stacks, 0AYR]}. \tag{3.6}$$

Lemma 3.4.1 ([BDP17, App. B.2 by B. Conrad]). *Let $\mathcal{X}$ be a connected proper algebraic stack of pure dimension 1 over a field k, and for $i = 1, 2$, let $q_i : U_i \to \mathcal{X}$ denote finite flat surjective morphisms with U_i a connected scheme. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$. Then*

$$\frac{\deg(q_1^*\mathcal{L})}{\deg q_1} = \frac{\deg(q_2^*\mathcal{L})}{\deg q_2}.$$

Proof. Since q_i is finite flat, $U_1 \times_{\mathcal{X}} U_2$ is a scheme. Let $U \subset U_1 \times_{\mathcal{X}} U_2$ be a connected component. Then U_1, U_2, U are all proper connected k-schemes of pure dimension 1. We have a diagram with all maps finite flat surjective

$$\begin{tikzcd} U \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & U_2 \arrow[d, "q_2"] \\ U_1 \arrow[r, "q_1"'] & \mathcal{X}. \end{tikzcd}$$

Noting that $\frac{\deg(p_i^*q_i^*\mathcal{L})}{\deg(q_i^*\mathcal{L})} = \deg p_i$ [Stacks, 0AYW,0AYZ], we find that the ratios in (3.6) are both equal to $\frac{\deg(p^*\mathcal{L})}{\deg p}$. $\square$

Definition 3.4.2 ([BDP17, App. B.2]). Let $\mathcal{X}$ be a connected proper algebraic stack of pure dimension 1 over a field k which admits a finite flat (equivalently, finite locally free [Stacks, 02KB]) surjective map $p : U \to \mathcal{X}$ with U a scheme. For a line bundle $\mathcal{L}$ on $\mathcal{X}$, define

$$\deg(\mathcal{L}) := \frac{\deg(p^*\mathcal{L})}{\deg p} \in \mathbb{Q}.$$

It follows from the lemma that $\deg(\mathcal{L})$ is independent of the choice of the finite flat scheme cover $p : U \to \mathcal{X}$. It will be useful to keep the following lemma in mind:

LEMMA 3.4.3 ([Knu71]). *Let X be a reduced algebraic space of pure dimension 1 which is separated and of finite type over a field k. Then X is a scheme. In particular, any 1-dimensional smooth separated Deligne-Mumford stack over a field k admits a coarse scheme.*

Proof. For k algebraically closed, X is a scheme by [Knu71, Th. V.4.9]. To bootstrap to general fields, one can use [Stacks, 0B88, 088J]. The last statement follows from the Keel-Mori Theorem 2.1.9. $\square$

Recall that a Deligne-Mumford stack is *tame* if its automorphism groups at every geometric point $\operatorname{Spec}\Omega \to \mathcal{X}$ have order coprime to the characteristic of Ω. We say it is generically tame if it has an open dense substack which is tame. The following proposition implies that it makes sense to speak of degrees of line bundles on $\mathcal{A}dm(G), \overline{\mathcal{M}(G)}$.

PROPOSITION 3.4.4 ([KV04], [EHKV01]). *We work over a field k.*

(a) *Let $\mathcal{X}$ be a smooth separated generically tame 1-dimensional Deligne-Mumford stack. Then there exists a finite flat surjective map $U \to \mathcal{X}$ with U a smooth k-scheme.*
(b) *Let G be a finite group. If $\operatorname{char}(k) \nmid 2|G|$, then for $\mathcal{M} = \mathcal{A}dm(G)_k$ or $\mathcal{M} = \overline{\mathcal{M}(G)}_k$, $\mathcal{M}$ admits a finite flat surjective map $U \to \mathcal{M}$ with U a smooth k-scheme.*

Proof. By Theorem 2.1.9 and Lemma 3.4.3, $\mathcal{X}$ admits a coarse scheme which is proper over k. Since proper schemes of dimension 1 are projective, (a) is [KV04, Ths. 1, 2, 3]. By Proposition 2.5.10(e), $\overline{\mathcal{M}(G)}$ is smooth with smooth projective coarse scheme. Since $\mathcal{A}dm(G) \to \overline{\mathcal{M}(G)}$ is an étale gerbe, the same is true of $\mathcal{A}dm(G)$. The assumption on $\operatorname{char}(k)$ guarantees that $\mathcal{A}dm(G)$ and $\overline{\mathcal{M}(G)}$ are generically tame, so (b) follows from (a). $\square$

Next we obtain a criterion to detect when $\deg(\mathcal{L})$ is an integer. Let $\mathcal{L}$ be an invertible sheaf on a Deligne-Mumford stack $\mathcal{L}$. For any geometric point $x : \operatorname{Spec}\Omega \to \mathcal{X}$, $x^*\mathcal{L}$ is a rank 1 representation of $\operatorname{Aut}_{\mathcal{X}}(x)$, which we call the *local character* of $\mathcal{L}$ at x. We will need the following result.

PROPOSITION 3.4.5 ([Ols12, Prop. 6.1]). *Let $\mathcal{X}$ be a locally finitely presented tame separated Deligne-Mumford stack admitting a coarse scheme $c : \mathcal{X} \to X$. Then the pullback c^* induces an isomorphism between the category of invertible sheaves on X and the full subcategory of invertible sheaves on $\mathcal{X}$ whose local characters at all geometric points are trivial. A quasi-inverse is given by c_*.*

Definition 3.4.6. Let $\mathcal{X}$ be an irreducible Deligne-Mumford stack admitting a coarse scheme $c : \mathcal{X} \to X$. Its *generic automorphism group* is the automorphism group of a geometric generic point $\operatorname{Spec}\Omega \to \mathcal{X}$.

PROPOSITION 3.4.7. *Let $\mathcal{X}$ be a connected tame smooth proper 1-dimensional Deligne-Mumford stack over a field k. Suppose its generic automorphism group has order n. Then for any invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ with trivial local characters, we have*

$$\deg(\mathcal{L}) = \frac{1}{n}\deg(c_*\mathcal{L}) \in \frac{1}{n}\mathbb{Z}.$$

Proof. By Proposition 3.4.4, we may find a finite flat map $p : Y \to \mathcal{X}$ with Y a smooth proper curve, so we may speak of degrees of line bundles on $\mathcal{X}$. By Lemma 2.1.10, X is a smooth proper curve, so the map $c \circ p : Y \to \mathcal{X} \to X$ is finite flat. By Proposition 3.4.5, $\mathcal{L} \cong c^*c_*\mathcal{L}$, so it would suffice to show that $\deg(c \circ p) = \frac{\deg p}{n}$. By the local structure of Deligne-Mumford stacks [Ols16, Th. 11.3.1], $\mathcal{X} \to X$ is étale-locally given by $[U/\Gamma] \to U/\Gamma$ for some finite group Γ acting on a scheme U étale over $\mathcal{X}$. Let $\Gamma_1 \le \Gamma$ be the kernel of the Γ-action on U. Then by shrinking U we may assume that U is irreducible and Γ/Γ_1 acts with trivial inertia on U. In this case it follows that Γ_1 is isomorphic to the generic automorphism group of $\mathcal{X}$, so $|\Gamma_1| = n$. We have a diagram

$$\begin{tikzcd}
Y \times_{\mathcal{X}} U \arrow[r, "\gamma"] \arrow[d, "\alpha"] & Y \times_{\mathcal{X}} [U/\Gamma] \arrow[r] \arrow[d, "\delta"] & Y \arrow[d, "p"] \\
U \arrow[r] \arrow[dr, "\beta"'] & {[U/\Gamma]} \arrow[r] \arrow[d, "\epsilon"] & \mathcal{X} \arrow[d, "c"] \\
& U/\Gamma \arrow[r] & X,
\end{tikzcd}$$

where all squares are cartesian. Thus γ is finite étale of degree $|\Gamma|$ and $\deg\alpha = \deg p$. Since Γ/Γ_1 acts with trivial inertia on U, β is finite étale of degree

$|\Gamma/\Gamma_1| = \frac{1}{n}|\Gamma|$. Since $\gamma \circ \delta \circ \epsilon = \beta \circ \alpha$ are finite flat, we have

$$\deg(c \circ p) = \deg(\epsilon \circ \delta) = \frac{\deg \alpha \cdot \deg \beta}{\deg \gamma} = \frac{\deg p \cdot \frac{1}{n}|\Gamma|}{|\Gamma|} = \deg p \cdot \frac{1}{n}$$

as desired. □

PROPOSITION 3.4.8. *We work over a field k.*

(a) *Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite flat map of connected proper 1-dimensional algebraic stacks of degree d, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then*
$$\deg(f^*\mathcal{L}) = d \cdot \deg(\mathcal{L}).$$

(b) *Let $\mathcal{X}$ be a connected tame smooth proper 1-dimensional Deligne-Mumford stack with generic automorphism group of order n. Let $c : \mathcal{X} \to X$ be its coarse scheme. Then for a line bundle $\mathcal{L}$ on X,*
$$\deg(c^*\mathcal{L}) = \frac{1}{n}\deg(\mathcal{L}).$$

(c) *Let $\mathcal{Y}, \mathcal{X}$ be connected tame smooth proper 1-dimensional Deligne-Mumford stacks with generic automorphism groups of order $n_{\mathcal{Y}}, n_{\mathcal{X}}$ respectively and admitting coarse schemes Y, X respectively. If $f : \mathcal{Y} \to \mathcal{X}$ induces a finite flat map $\overline{f} : Y \to X$ on coarse schemes, then for any line bundle $\mathcal{L}$ on $\mathcal{X}$, we have*
$$\deg(f^*\mathcal{L}) = \frac{n_{\mathcal{X}}}{n_{\mathcal{Y}}}\deg(\overline{f}) \cdot \deg(\mathcal{L}).$$

Proof. For (a), see [Stacks, 0AYW,0AYZ,02RH]. Part (b) follows from Propositions 3.4.5 and 3.4.7. For part (c), it follows from the étale local picture of Deligne-Mumford stacks [AV02, Lemma 2.2.3] that for some integer $m \geq 1$, $\mathcal{L}^{\otimes m}$ has trivial local characters. Since $\deg(\mathcal{L}^{\otimes m}) = m \cdot \deg(\mathcal{L})$, we are reduced to the case where $\mathcal{L}$ has trivial local characters, but in this case the result follows from (b). □

PROPOSITION 3.4.9. *Here we work over $\overline{\mathbb{Q}}$. Let $\mathcal{E}(1) \to \overline{\mathcal{M}(1)}$ be the universal family of elliptic curves, with zero section σ_O. Then the* Hodge bundle *$\lambda := \sigma_O^*\omega_{\mathcal{E}(1)/\overline{\mathcal{M}(1)}}$ is an invertible sheaf on $\overline{\mathcal{M}(1)}$ of degree $\frac{1}{24}$.*

Proof. Recall that if $\mathcal{M}$ is an algebraic stack and $\mathcal{F}$ is an $\mathcal{O}_{\mathcal{M}}$-module, its global sections is the set $\Gamma(\mathcal{M}, \mathcal{F}) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{O}_{\mathcal{M}}, \mathcal{F})$. Let $c : \overline{\mathcal{M}(1)} \to \overline{M(1)}$ be the coarse scheme. Thus, we have

$$\begin{aligned}\Gamma(\overline{\mathcal{M}(1)}, \lambda^{\otimes 12}) &= \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{M}(1)}}}(c^*\mathcal{O}_{\overline{M(1)}}, \lambda^{\otimes 12}) \\ &= \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{M}(1)}}}(\mathcal{O}_{\overline{M(1)}}, c_*\lambda^{\otimes 12}) = \Gamma(\overline{M(1)}, c_*\lambda^{\otimes 12}).\end{aligned}$$

Base changing to $\mathbb{C}$, by standard GAGA arguments,
$$\Gamma(\overline{\mathcal{M}(1)}, \lambda^{\otimes 12}) = \Gamma(\overline{M(1)}, c_*\lambda^{\otimes 12})$$
is isomorphic to the $\mathbb{C}$-vector space of modular forms of level 1 and weight 12, which has dimension 2, generated by the Eisenstein series E_{12} and the

discriminant Δ [DS05, Th. 3.5.2]. Since $\overline{M(1)} \cong \mathbb{P}^1$, it follows that $c_*\lambda^{\otimes 12}$ has degree 1, so by Proposition 3.4.8(b), $\deg(\lambda^{\otimes 12}) = \frac{1}{2}$, so $\deg(\lambda) = \frac{1}{24}$. $\square$

3.5. *The main congruence.* In this section we work universally over $\mathbb{S} = \operatorname{Spec} k$, where k is an algebraically closed field of characteristic 0.

THEOREM 3.5.1. *Let $\mathcal{X} \subset \mathcal{A}dm(G)$ be a connected component, with universal family $\mathcal{C} \xrightarrow{\pi} \mathcal{E} \to \mathcal{X}$ and reduced ramification divisor $\mathcal{R}_\pi$. Let $\mathcal{X} \to X$ be the coarse scheme of $\mathcal{X}$, and let $\overline{\mathcal{M}(1)} \to \overline{M(1)} \cong \mathbb{P}^1_j$ be the coarse moduli scheme. Let $\mathcal{E}(1) \to \overline{\mathcal{M}(1)}$ be the universal family. Let $\mathcal{R} \subset \mathcal{R}_\pi$ be a connected component with coarse scheme R, and let $\epsilon : \mathcal{R} \to \mathcal{X}$ be the induced map. By definition, $\mathcal{C}' := \mathcal{C} \times_{\mathcal{X}} \mathcal{R} \to \mathcal{R}$ admits a section σ lying over the zero section of $\mathcal{E}' := \mathcal{E} \times_{\mathcal{X}} \mathcal{R}$. To $\mathcal{X}$ we associate the three integers:*

- *let $e = e_{\mathcal{X}}$ be the ramification index of any point of C above the zero section of $\mathcal{E}$;*
- *let $d = d_{\mathcal{X}}$ be the degree of the induced map on coarse schemes $\overline{\epsilon} : R \to X$;*
- *let $m = m_{\mathcal{X}}$ be the minimal positive integer such that $(\sigma^*\Omega_{\mathcal{C}'/\mathcal{R}})^{\otimes m}$ has trivial local characters.*

Let $\overline{f} : X \to \overline{M(1)}$ be the map on coarse schemes induced by $f : \mathcal{X} \to \overline{\mathcal{M}(1)}$. Then we have

$$\deg(\overline{f}) \equiv 0 \mod \frac{12e}{\gcd(12e, md)}.$$

Proof. We have a commutative diagram with the top three squares cartesian (if you ignore the sections),

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "\pi'"] & \mathcal{C} \arrow[d, "\pi"] & \\
\mathcal{E}' \arrow[r] \arrow[d, shift right] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}(1) \arrow[d, shift right] \\
\mathcal{R} \arrow[r, "\epsilon"] \arrow[d] \arrow[u, shift right, "\sigma'_O"'] \arrow[uu, bend left=50, "\sigma"] & \mathcal{X} \arrow[r, "f"] \arrow[d] & \overline{\mathcal{M}(1)} \arrow[d] \arrow[u, shift right, "\sigma_O"'] \\
R \arrow[r, "\overline{\epsilon}"] & X \arrow[r, "\overline{f}"] & \overline{M(1)},
\end{tikzcd}$$

where σ_O, σ'_O denote the zero sections. The sheaf $\lambda := \sigma_O^*\omega_{\mathcal{E}(1)/\overline{\mathcal{M}(1)}}$ is the Hodge bundle, which has degree $\frac{1}{24}$ by Proposition 3.4.9. Since dualizing sheaves commute with arbitrary base change, by Proposition 3.3.1 we have

$$\epsilon^* f^* \lambda \cong \sigma_O'^{*}\omega_{\mathcal{E}'/\mathcal{R}} \cong (\sigma^*\omega_{\mathcal{C}'/\mathcal{R}})^{\otimes e} \cong (\sigma^*\Omega_{\mathcal{C}'/\mathcal{R}})^{\otimes e},$$

where the final isomorphism follows from the fact that σ lies in the smooth locus by definition of admissible covers. Suppose $\mathcal{R}$ has a generic automorphism group of order n. By Proposition 3.4.7, we have $\deg(\epsilon^* f^* \lambda) \in \frac{e}{mn}\mathbb{Z}$. Since $\deg(\lambda) = \frac{1}{24}$, by 3.4.8(c) we have

$$\deg(\epsilon^* f^* \lambda) = \frac{2}{n} \deg(\overline{\epsilon}) \cdot \deg(\overline{f}) \cdot \frac{1}{24} = \frac{d}{12n} \cdot \deg(\overline{f}) \in \frac{e}{mn}\mathbb{Z}$$

and hence

$$\deg(\overline{f}) \in \frac{12e}{dm}\mathbb{Z} \quad \text{or equivalently} \quad \deg(\overline{f}) \equiv 0 \mod \frac{12e}{\gcd(12e, md)}. \qquad \square$$

Theorem 3.5.1 takes as input a group G and a component $\mathcal{X} \subset \mathcal{A}dm(G)$, and outputs a congruence which at best gives $\equiv 0 \mod 12e_{\mathcal{X}}$, but is possibly diluted by the integers $d_{\mathcal{X}}, m_{\mathcal{X}}$ associated to $\mathcal{X}$. To obtain a good congruence, one wishes to show that $e_{\mathcal{X}}$ does not share large divisors with $d_{\mathcal{X}}$ and $m_{\mathcal{X}}$. Suppose $\mathcal{X}$ classifies covers with Higman invariant equal to the conjugacy class of $c \in G$. As we saw in Section 2.3, $e_{\mathcal{X}}$ is just the order of c. Here we will describe some ways to control $d_{\mathcal{X}}$ and $m_{\mathcal{X}}$.

By Proposition 3.2.2, $d_{\mathcal{X}}$ must divide the order of $C_G(\langle c\rangle)/\langle c\rangle$. A trivial consequence of this is that if ℓ^r is a prime power dividing $|c|$ with $\ell^{r+1} \nmid |G|$, then $\ell \nmid d_{\mathcal{X}}$. The integer $m_{\mathcal{X}}$ is more difficult to control. If $m'_{\mathcal{X}}$ denotes the minimum positive integer required to kill all the vertical automorphism groups of geometric points of $\mathcal{R}$ (Definition 2.5.5), then $12m'_{\mathcal{X}}$ kills all the automorphism groups of geometric points, so we must have $m_{\mathcal{X}} \mid 12m'_{\mathcal{X}}$. In fact, it follows from the local structure of admissible covers that the local characters of $\sigma^*\Omega_{\mathcal{C}'/\mathcal{R}}$ restrict to *faithful representations* of the vertical automorphism groups, so we have $m'_{\mathcal{X}} \mid m_{\mathcal{X}} \mid 12m'_{\mathcal{X}}$.

Let $\overline{r} : \operatorname{Spec}\Omega \to \mathcal{R}$ be a geometric point with image $\overline{x} \in \mathcal{X}$. Then $\overline{x}$ is given by a 1-generalized elliptic curve E over Ω and an admissible G-cover $\pi : C \to E$, and $\overline{r}$ is given by π together with a point $P \in C(\Omega)$ lying over $O \in E$. The vertical automorphism group of $\overline{x}$ is the group of G-equivariant automorphisms $\sigma \in \operatorname{Aut}(C)$ such that $\pi\circ\sigma = \pi$, and the vertical automorphism group of $\overline{r}$ is the subgroup consisting of vertical automorphisms of $\overline{x}$ satisfying $\sigma(P) = P$.

When E is smooth, there is a simple description of the vertical automorphism group of $\overline{r}$:

Proposition 3.5.2. *Let $\mathcal{Y} \subset \mathcal{A}dm^0(G)$ be a connected component classifying covers with Higman invariant $\mathfrak{c}$. Let $c \in \mathfrak{c}$ be a representative. Let $\mathcal{R}$ be a component of the ramification divisor of the universal family over $\mathcal{Y}$. Let $\overline{r}$ be a geometric point of $\mathcal{R}$ with image $\overline{y}$ in $\mathcal{Y}$. Suppose $\overline{r}$ corresponds to the admissible cover $\pi : C \to E$ together with the point $P \in \pi^{-1}(O)$. Then the vertical automorphism groups of $\overline{y}, \overline{r}$ are*

$$\operatorname{Aut}^v(\overline{y}) = Z(G) \qquad \text{and} \qquad \operatorname{Aut}^v(\overline{r}) = Z(G) \cap \langle c\rangle$$

Proof. The fact that $\mathrm{Aut}^v(\overline{y}) = Z(G)$ is Proposition 2.5.6. The subgroup of G-equivariant automorphisms which fix P is then $Z(G) \cap G_P$, where $G_P := \mathrm{Stab}_G(P)$ is the stabilizer. By the definition of the Higman invariant, G_P is conjugate to $\langle c \rangle$, so we have $\mathrm{Aut}^v(\overline{r}) = Z(G) \cap G_P = Z(G) \cap \langle c \rangle$. □

If E is not smooth, then C can fail to be irreducible, and the situation can be potentially bad enough to make the congruence trivial (see Section 4.11). The main purpose of the next section is to give a precise group-theoretic characterization of the vertical automorphism groups of cuspidal objects of $\mathcal{A}dm(G)$. This will allow us to control $m_{\mathcal{X}}$ at least when $G = \mathrm{SL}_2(\mathbb{F}_q)$ or a non-abelian finite simple group. In particular, we will show that in these cases, one can often achieve non-trivial congruences (see Section 4.10).

4. Galois correspondence for cuspidal objects of $\mathcal{A}dm(G)$

In this section we characterize cuspidal admissible G-covers combinatorially in terms of group-theoretic information in G and describe their automorphism groups. This is done in Section 4.10, though it will need terminology from the preceding subsections. We will also formulate a combinatorial version of Theorem 3.5.1 (Theorem 4.10.5), with which we will show that we can often obtain non-trivial congruences when G is a non-abelian group (Corollary 4.12.5). The statements of these two results can be understood without consulting the previous subsections.

We give an overview of our approach. We begin in Section 4.1 by defining the notion of a "precuspidal G-cover" of a non-smooth 1-generalized elliptic curve E, which is roughly a non-smooth admissible G-cover which may fail to be balanced or connected. The category of such objects is denoted $\mathcal{C}_E^{pc}$. Given a precuspidal G-cover $\pi : C \to E$, taking normalizations we obtain a G-cover $\pi' : C' \to \mathbb{P}^1$, ramified only above three points. We may assume that $0, \infty \in \mathbb{P}^1$ are the preimages of the node in E. The normalization map also provides the data of a G-equivariant bijection $\alpha = \alpha_\pi : \pi'^{-1}(0) \xrightarrow{\sim} \pi'^{-1}(\infty)$. We will let $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ denote the category of such pairs (π', α). The usual Galois correspondence identifies G-covers of $\mathbb{P}^1$ only branched over $\{0, 1, \infty\}$ with finite sets equipped with commuting actions of $\Pi := \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ and G. Accordingly, objects of $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ can be identified with finite sets equipped with commuting actions of Π and G as well as a "combinatorial G-equivariant bijection." The category of such objects is denoted $\underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$, where δ denotes a path from a tangential base point at $0 \in \mathbb{P}^1$ to a tangential base point at $\infty \in \mathbb{P}^1$. We will describe equivalences of categories

$$\mathcal{C}_E^{pc} \xrightarrow{\Xi} \mathcal{C}_{\mathbb{P}^1}^{\succ} \xrightarrow{F_\delta^{\succ}} \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}.$$

Roughly speaking, the first is given by taking normalizations, and the second is given by taking geometric fibers. We say that an object of $\mathcal{C}_E^{pc}$ is *cuspidal* if it corresponds to a cuspidal object of $\mathcal{A}dm(G)$. This amounts to the two additional conditions that the cover is *connected* and (the G-action is) *balanced*. The full subcategory of cuspidal objects of $\mathcal{C}_E^{pc}$ is denoted $\mathcal{C}_E^c$. To obtain a combinatorial characterization of $\mathcal{C}_E^c$, we must describe what it means for an object of $\underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ to come from a cuspidal (i.e., connected and balanced) object of $\mathcal{C}_E^{pc}$. This is done in Section 4.7. Using this, in Section 4.8 we give a combinatorial parametrization of all cuspidal objects of $\mathcal{A}dm(G)$ in terms their "δ-invariant" (where δ is the path mentioned above), which is an equivalence class of a generating pair of G. In Section 4.10, we calculate the automorphism groups of cuspidal admissible G-covers in terms of their δ-invariants, and in Section 4.12 we describe some applications to the cardinalities of Nielsen equivalence classes of generating pairs of finite groups.

4.1. *(Pre)cuspidal G-curves*, *(pre)cuspidal G-covers*. Throughout Section 4, we will work universally over $\mathbb{S} = \operatorname{Spec} k$, where k denotes an algebraically closed field of characteristic 0. Moreover, we will fix a compatible system of primitive n-th roots of unity $\{\zeta_n\}_{n\geq 1} \subset k$, compatible in the sense that for all $d \mid n$, $\zeta_n^d = \zeta_{n/d}$. Essentially all of our methods are algebraic, so the same development should also make sense in all tame characteristics.

Definition 4.1.1. A *precuspidal G-curve* is a non-smooth prestable curve C equipped with a faithful right action of G and a G-invariant divisor $R \subset C$ finite étale over k, such that the quotient map $C \to C/G$ sends nodes to nodes and is étale on $C_{\mathrm{sm}} - R$, and $(C/G, R/G)$ is a nodal elliptic curve (i.e., a non-smooth 1-generalized elliptic curve). A *cuspidal G-curve* is a precuspidal G-curve C such that

- C is *connected*;
- the G-action is *balanced* at the nodes in the sense of Remark 2.1.5(e) — in this case we say C is balanced.

A morphism of (pre)cuspidal G-curves is a G-equivariant map preserving divisors.

Note that up to isomorphism, there is only one nodal elliptic curve.

Definition 4.1.2. A G-cover of a finite type equidimension 1 scheme Y is a finite flat map $p : X \to Y$ equipped with a faithful right action of G on X such that p induces an isomorphism $X/G \xrightarrow{\sim} Y$.

Definition 4.1.3. Let $E = (E, O)$ be nodal elliptic curve. A *precuspidal G-cover* of E is a G-cover $\pi : C \to E$ such that $(C, (C \times_E O)_{\mathrm{red}})$ is a precuspidal G-curve. Equivalently, it is a G-cover $\pi : C \to E$ satisfying

- π sends nodes to nodes,
- π is étale on $C_{\mathrm{sm}} - \pi^{-1}(O)$

A precuspidal G-cover is *connected* (resp. *balanced*) if C is connected (resp. the G-action is balanced at the nodes). A *cuspidal G-cover* is a balanced connected precuspidal G-cover. A morphism of (pre)cuspidal G-covers is a G-equivariant map over E. Let $\mathcal{C}_E^{pc}$ (resp. $\mathcal{C}_E^c$) denote the category of precuspidal (resp. cuspidal) G-covers of E.

In particular, a precuspidal G-cover is an admissible G-cover if and only if it is cuspidal, or equivalently, connected and balanced, or equivalently, is a cuspidal object of $\mathcal{A}dm(G)$. Here is a precise statement.

PROPOSITION 4.1.4. *Let $\mathcal{A}dm^c(G) := \mathcal{A}dm(G) - \mathcal{A}dm^0(G)$ be the closed substack consisting of cuspidal (i.e., non-smooth) objects. Associating a cuspidal G-curve (C, R) to the cuspidal G-cover $C \to C/G$ of the nodal elliptic curve $(C/G, R/G)$ gives an equivalence of categories*

$$\{\text{cuspidal } G\text{-curves}\} \xrightarrow{\sim} \mathcal{A}dm^c(G)(k).$$

Let E be a nodal elliptic curve. Let $\mathcal{A}dm(G)_E$ denote the fiber category of $\mathcal{A}dm(G)$ over $E \in \overline{\mathcal{M}(1)}$. Thus, the objects of $\mathcal{A}dm(G)_E$ are admissible G-covers of E, and morphisms are morphisms of admissible G-covers which induce the identity on E. Then we have an equality of categories

$$\mathcal{C}_E^c = \mathcal{A}dm(G)_E.$$

In particular, the automorphism groups of objects of $\mathcal{C}_E^c$ (resp. $\mathcal{A}dm(G)_E$) are the vertical automorphism groups of the corresponding objects of $\mathcal{A}dm(G)$.

Proof. The first equivalence follows from Theorem 2.4.9, which in turn implies $\mathcal{C}_E^c = \mathcal{A}dm(G)_E$. □

4.2. *Tangential base points.* Here we follow Deligne [Del89, §15]. Let X be a smooth curve (a smooth finite type 1-dimensional scheme over $k = \overline{k}$), $x \in X$ a closed point, and $X^\circ := X - \{x\}$. Let $K_x := \operatorname{Frac} \mathcal{O}_{X,x}$, and let $v_x : K_x \to \mathbb{Z}$ be the discrete valuation. Then K_x is filtered by v_x:

$$F^i K_x := \{f \in K_x \mid v_x(f) \geq i\}.$$

As a scheme, define the tangent space at x by $\mathbb{T}_x := \operatorname{Spec} \operatorname{gr}(\mathcal{O}_{X,x})$, where $\operatorname{gr}(\mathcal{O}_{X,x})$ is the associated graded ring with respect to the filtration F^i defined above.[26] Similarly, the punctured tangent space $\mathbb{T}_x^\circ := \mathbb{T}_x - \{0\}$ is $\operatorname{Spec} \operatorname{gr}(K_x)$. These constructions are functorial in (X, x). If $\pi^\circ : Y^\circ \to X^\circ$ is a finite étale map which extends to a finite map $\pi : Y \to X$ of smooth curves, then we may

[26]Strictly speaking, this is really the tangent *cone*. In particular, we are viewing $\mathbb{T}_x$ as a scheme instead of as a vector space.

associate to π° the maps

$$\pi_{(x)} : Y_{(x)} := \coprod_{y\in\pi^{-1}(x)} \operatorname{Spec}\operatorname{gr}(\mathcal{O}_{Y,y}) \longrightarrow \operatorname{Spec}\operatorname{gr}(\mathcal{O}_{X,x}) =: \mathbb{T}_x,$$
$$\pi^\circ_{(x)} : Y^\circ_{(x)} := \coprod_{y\in\pi^{-1}(x)} \operatorname{Spec}\operatorname{gr}(K_y) \longrightarrow \operatorname{Spec}\operatorname{gr}(K_x) =: \mathbb{T}^\circ_x$$

defined as follows. The rings $\mathcal{O}_{Y,y}$ and $K_y := \operatorname{Frac}\mathcal{O}_{Y,y}$ are filtered according to the valuation $v_y : K_y \to \frac{1}{e}\mathbb{Z}$ extending v_x, where e is the ramification index of π at y. Specifically, for $i \in \frac{1}{e}\mathbb{Z}$,

$$F^iK_y = \{f \in K_y \mid v_y(f) \geq i\}.$$

If we also view K_x as a $\frac{1}{e}\mathbb{Z}$-filtered ring with $F^iK_x := F^iK_y \cap K_x$ for $i \in \frac{1}{e}\mathbb{Z}$, then $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and $K_x \to K_y$ are filtered ring maps, and we define the maps $\pi_{(x)}, \pi^\circ_{(x)}$ to be the maps which are induced by these filtered ring maps. Let $\varpi_x \in \mathcal{O}_{X,x}$ be a uniformizer, and let $\varpi_y \in \mathcal{O}_{Y,y}$ satisfy $\varpi_y^e \equiv \varpi_x \mod \mathfrak{m}_y^{e+1}$, so that ϖ_y is a uniformizer of $\mathcal{O}_{Y,y}$. Then the maps

$$\begin{aligned} k[t] &\longrightarrow \operatorname{gr}(\mathcal{O}_{X,x}), & k[s] &\xrightarrow{\sim} \operatorname{gr}(\mathcal{O}_{Y,y}),\\ t &\mapsto \varpi_x \mod \mathfrak{m}_x = F^1\mathcal{O}_{X,x}, & s &\mapsto \varpi_y \mod \mathfrak{m}_y = F^{1/e}\mathcal{O}_{Y,y} \end{aligned}$$

are isomorphisms [Stacks, 00NO] which moreover induce isomorphisms $k[t,t^{-1}] \cong \operatorname{gr}(K_x)$ and $k[s,s^{-1}] \cong \operatorname{gr}(K_y)$. Since $\varpi_y^e \equiv \varpi_x \mod \mathfrak{m}_y^{e+1}$, the restriction of $\pi^\circ_{(x)}$ to $\operatorname{Spec}\operatorname{gr}(K_y)$ fits into a commutative diagram

$$\begin{tikzcd} k[s,s^{-1}] \arrow[r, "\cong"] & \operatorname{gr}(K_y). \\ k[t,t^{-1}] \arrow[u, "t\mapsto s^e"] \arrow[r, "\cong"] & \operatorname{gr}(K_x) \arrow[u, "\pi^\circ_{(x)}"'] \end{tikzcd}$$

In particular, we find that $\pi^\circ_{(x)}$ is finite étale. For a scheme S, let FEt_S denote the category of finite étale maps to S. Then the association $\pi^\circ \mapsto \pi^\circ_{(x)}$ defines an exact functor[27] $R_x : \mathrm{FEt}_{X^\circ} \to \mathrm{FEt}_{\mathbb{T}^\circ_x}$, and hence if $t \in \mathbb{T}^\circ_x$ is a closed point with associated fiber functor $F_{\mathbb{T}^\circ_x,t}$, then the composition

$$F_{X^\circ,t} : \mathrm{FEt}_{X^\circ} \xrightarrow{R_x} \mathrm{FEt}_{\mathbb{T}^\circ_x} \xrightarrow{F_{\mathbb{T}^\circ_x,t}} \underline{\mathbf{Sets}}$$

is also a fundamental functor for FEt_{X° [GR71, Exp. V, Prop. 6.1]. That is to say, if $\Pi := \operatorname{Aut}(F_{X^\circ,t})$, then $F_{X^\circ,t}$ defines an equivalence of categories between FEt_{X° and the category of finite sets with Π-action.

[27]That is, it preserves finite limits and finite colimits. This can be checked using [GR71, Exp. V, Prop. 6.1].

Definition 4.2.1. Let X be a smooth curve, $x \in X$ be a closed point, and let $X^\circ := X - \{x\}$. A *tangential base point* of X at x is a closed point $t \in \mathbb{T}_x^\circ$. Given a finite étale cover $\pi : Y^\circ \to X^\circ$, we will abuse notation and write

$$Y_t^\circ = \pi^{-1}(t) := F_{X^\circ,t}(\pi)$$

and call $F_{X^\circ,t}$ the *fiber of* $\pi : Y^\circ \to X^\circ$ *above* t. Accordingly, we will write $\pi_1(X^\circ, t) := \mathrm{Aut}(F_{X^\circ,t})$.

The group $\pi_1(\mathbb{T}_x^\circ, t) := \mathrm{Aut}(F_{\mathbb{T}_x^\circ,t}) \cong \widehat{\mathbb{Z}}$ acts on $F_{X^\circ,t}$ in the obvious way, and this defines a canonical morphism of fundamental groups

$$\pi_1(\mathbb{T}_x^\circ, t) \to \pi_1(X^\circ, t),$$

which is in fact injective [GR71, V, Prop. 6.8]. There is a unique isomorphism $\mathbb{T}_x \xrightarrow{\sim} \mathbb{A}^1$ sending $0 \mapsto 0$ and $t \mapsto 1$ which induces an isomorphism $\mathbb{T}_x^\circ \xrightarrow{\sim} \mathbb{G}_m := \mathbb{A}^1 - \{1\}$ (which we will use to identify the two schemes). The Galois theory of $\mathbb{G}_m$ is well-understood: for every integer $n \geq 1$, there is a unique connected degree n finite étale cover of $\mathbb{G}_m$ given by $\pi_n : \mathbb{G}_m \to \mathbb{G}_m$ sending $z \mapsto z^n$. Let $g_{t,n} \in \mathrm{Gal}(\pi_n)$ be given by $z \mapsto \zeta_n z$.

Definition 4.2.2. Let X be a smooth curve, $x \in X$ be a closed point, let $X^\circ := X - \{x\}$, and let $t \in \mathbb{T}_x^\circ$ be a tangential base point. The *canonical generator of inertia* is the element $\gamma_t \in \pi_1(\mathbb{T}_x^\circ, t)$ which acts on $F_{\mathbb{T}_x^\circ,t}(\pi_n)$ via $g_{t,n} : z \mapsto \zeta_n z$ for every n. We will often view γ_t as an element of $\pi_1(X^\circ, t)$ via the canonical map $\pi_1(\mathbb{T}_x^\circ, t) \hookrightarrow \pi_1(X^\circ, t)$. Note that this definition depends on our choice of compatible system $\{\zeta_n\}_{n\geq 1}$.

Proposition 4.2.3. *Let X be a smooth curve, $x \in X$ a closed point, and $X^\circ := X - \{x\}$.*

(a) *Let T_x^*X be the Zariski cotangent space at x (a vector space), and let $T_0^*\mathbb{T}_x$ be the Zariski cotangent space at $0 \in \mathbb{T}_x$. There is a canonical isomorphism*

$$T_x^*X \xrightarrow{\sim} T_0^*\mathbb{T}_x$$

which is functorial in (X, x).

(b) *Let $\pi : Y \to X$ be a finite map of smooth curves, étale over X°. Every connected component of the scheme $Y_{(x)}$ has a unique point lying over $0 \in \mathbb{T}_x$. The fiber $\pi_{(x)}^{-1}(0)$ is canonically in bijection with $Y_x := \pi^{-1}(x)$. The map $Y_t := \pi_{(x)}^{-1}(t) \to \pi_{(x)}^{-1}(0)$ sending $y \in Y_t$ to the unique point of $Y_{(x)}$ lying over 0 induces a canonical bijection (the specialization to a ramified fiber)*

$$\langle\gamma_t\rangle\backslash Y_t \xrightarrow{\sim} \pi_{(x)}^{-1}(0) \xrightarrow{\sim} Y_x$$

which is functorial in π, in the sense that it defines an isomorphism of functors $\mathrm{FEt}_{X^\circ} \to \underline{\mathbf{Sets}}$. Given $y \in Y_t$, we will often let $[x]$ denote its image in $\langle\gamma_t\rangle\backslash Y_t$ or its image in Y_x.

Proof. For (a), to be precise let $\mathfrak{m}_0 \subset \mathrm{Spec}\,\mathrm{gr}(\mathcal{O}_{X,x})$ be the maximal ideal at 0. Then for $r \geq 0$, $\mathfrak{m}_0^r = \oplus_{n\geq r}\mathrm{gr}_n(\mathcal{O}_{X,x})$, so the Zariski cotangent space

$\mathfrak{m}_0/\mathfrak{m}_0^2$ is canonically isomorphic to $\mathrm{gr}_1(\mathcal{O}_{X,x}) = \mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^*X$, and this isomorphism is clearly functorial in the pair (X, x).

For (b), it suffices to observe that from the discussion above, the scheme $Y_{(x)} := \sqcup_{y\in\pi^{-1}(x)} \operatorname{Spec}\mathrm{gr}(\mathcal{O}_{Y,y})$ is a disjoint union of affine lines indexed by $y \in \pi^{-1}(x)$, each mapping to $\mathbb{T}_x$ (also an affine line) by the map $z \mapsto z^{e_y}$, where e_y is the ramification index of π at y. □

Remark 4.2.4. If $k = \mathbb{C}$, there is a complementary analytic theory [Del89, §15.3-15.12], where the functor $R_x : \mathrm{FEt}_{X^\circ} \to \mathrm{FEt}_{\mathbb{T}_x^\circ}$ is given by pulling back covers along the germ of a local isomorphism between a punctured neighborhood of $0 \in \mathbb{T}_x^\circ$ and a punctured neighborhood of $x \in X$. As an informal picture, one should imagine gluing a copy of $\mathbb{C}^\times$ to X° along a (infinitesimally) small punctured disk at 0 and x. In this picture a tangential base point is simply a point of $\mathbb{C}^\times$. The resulting space is homeomorphic to X°, and hence has the same fundamental group. Moreover, in this analytic theory, there is a good notion of "path" in the topological sense given by continuous maps from $[0, 1]$, and so one can speak of true paths between points (normal, or tangential).

For general algebraically closed k of characteristic 0, one can often deduce results from the analytic theory over $\mathbb{C}$ as follows. First note that depending on the cardinality of k, either k embeds in $\mathbb{C}$ or it contains $\mathbb{C}$. On the other hand, if L/K is an extension of algebraically closed fields and X is a K-scheme, then the map $X_L \to X$ induces an isomorphism on fundamental groups for any choice of base points [GR71, Exp. XIII, Prop. 4.6], so it induces an equivalence $\mathrm{FEt}_X \cong \mathrm{FEt}_{X_L}$. Finally, classical GAGA results [GR71, Exp. XII] provide an equivalence between the category of finite étale covers of finite type $\mathbb{C}$-schemes and finite étale covers of their analytifications.

Remark 4.2.5. There is a simpler alternative version of tangential base points given as follows. Let X be a smooth curve over k, and let $x \in X$ be a closed point. Let $\varpi \in \mathcal{O}_{X,x}$ be a uniformizer. Then the map $\mathcal{O}_{X,x} \to k[\![t]\!]$ sending $\varpi \mapsto t$ defines a map

$$t_\varpi : \operatorname{Spec} k(\!(z^{1/\infty})\!) \xrightarrow{i_z} \operatorname{Spec} k(\!(z)\!) \to \operatorname{Spec} k[\![z]\!] \xrightarrow{\sim} \operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \longrightarrow X$$

such that t_ϖ becomes a geometric point of X°. This gives a true geometric point of X° and shares many of the same features as the tangential base points given above (canonical generators of inertia, specialization map to a ramified fiber). However in this definition, it is more difficult to describe the corresponding analytic theory over $\mathbb{C}$, which we will need in two places.

4.3. *The fundamental group of* $\mathbb{P}^1 - \{0, 1, \infty\}$. Here we set up some of the notation and convention which we will maintain for the rest of Section 4.

Let $\mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}$. Recall that the fundamental groupoid of $\mathbb{P}^*$ is the category $\Pi_{\mathbb{P}^*}$ whose objects are fundamental functors $\mathrm{FEt}_{\mathbb{P}^*} \to \underline{\mathbf{Sets}}$

(see [GR71, §V, Def. 5.1]) and whose morphisms are isomorphisms of functors. If $x \in \mathbb{P}^*$ is a geometric point or tangential base point, let F_x denote the corresponding fundamental functor, which we also call the *fiber functor at* x. The fundamental group of $\mathbb{P}^*$ with base point x is the group $\pi_1(\mathbb{P}^*, x) := \mathrm{Aut}(F_x)$. If x, y are geometric points or tangential base points, then a path $\delta : x \rightsquigarrow y$ is an isomorphism of fiber functors $\delta : F_x \xrightarrow{\sim} F_y$. If $\delta : x \rightsquigarrow y$ and $\delta' : y \rightsquigarrow z$ are paths, then we will write the composition $\delta' \circ \delta$ as $\delta'\delta$ or $\delta' \cdot \delta : x \rightsquigarrow z$. If $\gamma \in \pi_1(\mathbb{P}^*, y) := \mathrm{Aut}(F_y)$ and $\delta : x \rightsquigarrow y$ is a path, then we write "conjugation" as

$$\gamma^\delta := \delta^{-1}\gamma\delta \in \pi_1(\mathbb{P}^*, x) := \mathrm{Aut}(F_x). \tag{4.1}$$

An automorphism $f \in \mathrm{Aut}(\mathbb{P}^*)$ determines by pullback an automorphism $f^* : \mathrm{FEt}_{\mathbb{P}^*} \xrightarrow{\sim} \mathrm{FEt}_{\mathbb{P}^*}$ which induces an automorphism of the fundamental groupoid $f_* : \Pi_{\mathbb{P}^*} \xrightarrow{\sim} \Pi_{\mathbb{P}^*}$ described by $f_*F_x := F_x \circ f^*$. There is a canonical isomorphism $F_x \circ f^* \cong F_{f(x)}$, using which we obtain the usual induced map $f_* : \pi_1(\mathbb{P}^*, x) \to \pi_1(\mathbb{P}^*, f(x))$.

Let $\iota : z \mapsto \frac{1}{z}$ denote the unique automorphism of $\mathbb{P}^1$ fixing 1 and switching $0, \infty$. In what follows, we will fix a choice of tangential base point t_0 at $0 \in \mathbb{P}^1$ and define t_∞ as the tangential base point at ∞ given by $\iota(t_0)$. If γ_0, γ_∞ denote the canonical generators of inertia at t_0, t_∞, then ι defines a homomorphism $\iota_* : \pi_1(\mathbb{P}^*, t_0) \to \pi_1(\mathbb{P}^*, t_\infty)$ sending $\gamma_0 \mapsto \gamma_\infty$. If $\delta : t_0 \rightsquigarrow t_\infty$ is a path, then $\iota_*\delta$ is a path $t_\infty \rightsquigarrow t_0$.

Definition 4.3.1. Having fixed a tangential base point t_0 at 0, let $t_\infty := \iota(t_0)$. A path $\delta : t_0 \rightsquigarrow t_\infty$ is *good* if

- γ_0 and $\gamma_\infty^\delta := \delta^{-1}\gamma_\infty\delta$ topologically generate $\pi_1(\mathbb{P}^*, t_0)$; and
- for some path $\epsilon : t_0 \rightsquigarrow t_1$, we have $\gamma_\infty^\delta\gamma_1^\epsilon\gamma_0 = 1$.

The path δ is *symmetric* if

$$\iota_*\delta = \gamma_0^r \cdot \delta^{-1} \cdot \gamma_\infty^s \qquad \text{for some } r, s \in \mathbb{Z}.$$

It follows from the analytic theory (cf. Remark 4.2.4) that symmetric good paths exist. Indeed, we can let δ be the path $t_0 \rightsquigarrow t_\infty$ in $\mathbb{P}^*$ which, for some small $\epsilon > 0$, traces the interval $(0, 1-\epsilon)$, makes a small counterclockwise turn around $1 \in \mathbb{P}^1$, and continues along the interval $(1+\epsilon, \infty)$.

4.4. *The category $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ and the normalization map $\Xi : \mathcal{C}_E^{pc} \longrightarrow \mathcal{C}_{\mathbb{P}^1}^{\succ}$.* As in Section 4.3, in what follows let $\mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}$. We fix a choice of tangential base point t_0 at 0. Let $t_\infty := \iota(t_0)$ denote the corresponding tangential base point at ∞ (see Section 4.3). Then associated to t_0, t_∞ we have fiber functors

$$F_{t_0}, F_{t_\infty} : \mathrm{FEt}_{X^\circ} \longrightarrow \underline{\mathbf{Sets}}$$

and canonical generators of inertia $\gamma_0 \in \pi_1(\mathbb{P}^*, t_0), \gamma_\infty \in \pi_1(\mathbb{P}^*, t_\infty)$.

Definition 4.4.1. Let $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ be the category of pairs $(q : D \to \mathbb{P}^1, \alpha)$, where q is a (possibly disconnected) smooth G-cover, étale over $\mathbb{P}^*$, and α is a G-equivariant bijection of fibers $\alpha : D_0 \to D_\infty$, where $D_0 := q^{-1}(0)$ and $D_\infty := q^{-1}(\infty)$. A morphism between $(q : D \to \mathbb{P}^1, \alpha) \to (q' : D' \to \mathbb{P}^1, \alpha')$ in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ is a G-equivariant morphism $f : D \to D'$ over $\mathbb{P}^1$ respecting the identifications of the fibers. In a formula, we require that f satisfies

$$f(\alpha(x)) = \alpha'(f(x)) \qquad \text{for all } x \in D_0$$

Informally, $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ is the category of three point covers with "gluing data." The symbol "$\succ$" represents that the objects come with gluing data.

Definition 4.4.2. Given a nodal elliptic curve E, a *standard normalization* of E is a finite birational map $\nu : \mathbb{P}^1 \to E$ satisfying

- $\nu(1) = O \in E$;
- $\nu|_{\nu^{-1}(E^{\mathrm{sm}})} : \nu^{-1}(E^{\mathrm{sm}}) \to E^{\mathrm{sm}}$ is an isomorphism;
- let $z \in E$ be the node — then $\nu^{-1}(z) = \{0, \infty\} \subset \mathbb{P}^1$.

The map $\nu : \mathbb{P}^1 \to E$ identifies $\mathbb{P}^1$ with the normalization of E, and hence up to precomposing with $\iota : z \mapsto 1/z$, the map ν is uniquely determined by these properties.

Let $\nu : \mathbb{P}^1 \to E$ be a standard normalization, and let $p : C \to E$ be a precuspidal G-cover. Let $\nu_C : C' \to C$ denote the normalization of C. Then the G-action on C extends uniquely to C', and there is a unique map $p' : C' \to \mathbb{P}^1$ fitting into the commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\nu_C} & C \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{\nu} & E \end{array}$$

such that p' is a G-cover. Note that C' may be disconnected even if C is connected. The map ν_C defines a G-equivariant bijection (a "gluing datum")

$$\alpha_p : C'_0 \xrightarrow{\sim} C'_\infty$$

defined by sending $x \in C'_0$ to the unique point $y \in C'_\infty$ such that $\nu_C(x) = \nu_C(y)$. Let $C'^\circ := p'^{-1}(\mathbb{P}^*)$. Then the restriction $p'|_{C'^\circ} : C'^\circ \to \mathbb{P}^*$ is étale, and so the pair (p', α_p) defines an object of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Since a morphism of objects in $\mathcal{C}^{pc}_E$ induces a morphism of their images in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$, taking normalizations and remembering the gluing data defines a functor

$$\Xi_\nu : \mathcal{C}^{pc}_E \longrightarrow \mathcal{C}^{\succ}_{\mathbb{P}^1},$$
$$(p : C \to E) \mapsto (p', \alpha_p).$$

If ν' is another standard normalization, then $\nu' = \nu \circ \iota$, and $\iota : z \mapsto 1/z$ induces an isomorphism of functors $\Xi_\nu \cong \Xi_{\nu'}$. We will eventually show that Ξ_ν is an equivalence of categories. Let $(q : D \to \mathbb{P}^1, \alpha)$ be an object of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. We summarize some of the relevant structures associated to (q, α).

(a) A path $\delta : t_0 \rightsquigarrow t_\infty$ defines a bijection of fibers

$$\delta(q) : D_{t_0} \xrightarrow{\sim} D_{t_\infty}$$

which is G-equivariant since δ is an isomorphism of functors. Sometimes we will abuse notation and simply write $\delta = \delta(q)$. Moreover, it determines an isomorphism of fundamental groups:

$$\begin{aligned} {}^{\delta}(\cdot) : \pi_1(\mathbb{P}^*, t_0) &\xrightarrow{\sim} \pi_1(\mathbb{P}^*, t_\infty), \\ \gamma &\mapsto {}^{\delta}\gamma := \delta\gamma\delta^{-1}. \end{aligned}$$

If we let $\pi_1(\mathbb{P}^*, t_0)$ act on D_{t_∞} via this isomorphism, then $\delta(q)$ is $\pi_1(\mathbb{P}^*, t_0)$-equivariant.

(b) For any point $x_0 \in D_{t_0}$, the freeness and transitivity of the G-action on fibers of $q|_{q^{-1}(\mathbb{P}^*)}$ yield monodromy representations

$$\begin{aligned} \varphi_{x_0} : \pi_1(\mathbb{P}^*, t_0) &\longrightarrow G, \\ \varphi_{\delta x_0} : \pi_1(\mathbb{P}^*, t_\infty) &\longrightarrow G \end{aligned}$$

defined in the usual way by the relations

$$\begin{array}{rcll} \gamma \cdot x_0 & = & x_0 \cdot \varphi_{x_0}(\gamma) & \text{for any } \gamma \in \pi_1(\mathbb{P}^*, t_0), \\ \gamma \cdot \delta x_0 & = & \delta x_0 \cdot \varphi_{\delta x_0}(\gamma) & \text{for any } \gamma \in \pi_1(\mathbb{P}^*, t_\infty). \end{array}$$

(c) Let $[\cdot]$ denote the map

$$\begin{aligned} [\cdot] : D_{t_0} &\longrightarrow \langle \gamma_0 \rangle \backslash D_{t_0}, \\ x &\mapsto [x] := \langle \gamma_0 \rangle x \end{aligned}$$

and similarly for points of D_{t_∞}.

(d) Let $\gamma_\infty^\delta := \delta^{-1}\gamma_\infty\delta$. As in Proposition 4.2.3, there are canonical bijections functorial in D

$$\begin{array}{rcl} \xi_0 : \langle \gamma_0 \rangle \backslash D_{t_0} & \xrightarrow{\sim} & D_0, \\ \xi_\infty : \langle \gamma_\infty \rangle \backslash D_{t_\infty} & \xrightarrow{\sim} & D_\infty, \\ \xi_\infty \circ \delta : \langle \gamma_\infty^\delta \rangle \backslash D_{t_0} & \xrightarrow{\sim} & D_\infty. \end{array}$$

Here, functoriality means the following. Let F_0 (resp. F_0') be the functors $\mathcal{C}^{\succ}_{\mathbb{P}^1} \to \underline{\mathbf{Sets}}$ sending $(q : D \to \mathbb{P}^1, \alpha)$ to D_0 (resp. $\langle \gamma_0 \rangle \backslash D_{t_0}$). Then ξ_0 defines a natural isomorphism of functors $\xi_0 : F_0' \xrightarrow{\sim} F_0$. The map $\xi_0 \circ [\cdot] : D_{t_0} \to D_0$ should be thought of as the "specialization map to the ramified fiber" induced by the "specialization" $t_0 \rightsquigarrow 0$ in $\mathbb{P}^1$. Thus for $x \in D_{t_0}$, we will often abuse notation and view $[x]$ as an element of D_0, and similarly for points of D_{t_∞}. Via ξ_0, ξ_∞, we will often view α as a bijection of coset spaces

$$\alpha : \langle \gamma_0 \rangle \backslash D_{t_0} \xrightarrow{\sim} \langle \gamma_\infty \rangle \backslash D_{t_\infty}.$$

(e) Given a point $x \in D_0$, let $G_x := \mathrm{Stab}_G(x)$. The right-action of G on D induces a left action on the cotangent space T_x^*D, whence a *local representation*

$$\chi_x : G_x \longrightarrow \mathrm{GL}(T_x^*D). \tag{4.2}$$

Next we record some basic computations which we will use freely in what follows. Recall that for a path $\delta : x \rightsquigarrow y$ and a loop γ at y, we write $\gamma^\delta := \delta^{-1}\gamma\delta$ to be the loop at x given by $x \overset{\delta}{\rightsquigarrow} y \overset{\gamma}{\rightsquigarrow} y \overset{\delta^{-1}}{\rightsquigarrow} x$ (see (4.1)).

PROPOSITION 4.4.3. *Let $(q : D \to \mathbb{P}^1, \alpha) \in \mathcal{C}^{\succ}_{\mathbb{P}^1}$, let $x_0 \in D_{t_0}$, and let $\delta : t_0 \rightsquigarrow t_\infty$ be a path. Then we have*

(a) $G_{[x_0]} = \langle \varphi_{x_0}(\gamma_0) \rangle$ *and* $\chi_{[x_0]}(\varphi_{x_0}(\gamma_0)) = \zeta_n$;
(b) $G_{[\delta x_0]} = \langle \varphi_{\delta x_0}(\gamma_\infty) \rangle$ *and* $\chi_{[\delta x_0]}(\varphi_{\delta x_0}(\gamma_\infty)) = \zeta_n$;
(c) $\varphi_{\delta x_0}(\gamma) = \varphi_{x_0}(\gamma^\delta)$ *for all* $\gamma \in \pi_1(\mathbb{P}^*, t_\infty)$;
(d) $\varphi_{\delta^{-1}x_\infty}(\gamma) = \varphi_{x_\infty}(\gamma^{\delta^{-1}})$ *for all* $\gamma \in \pi_1(\mathbb{P}^*, t_\infty)$, $x_\infty \in D_{t_\infty}$;
(e) $\varphi_{xg}(\gamma) = g^{-1}\varphi_x(\gamma)g$ *for all* $g \in G, \gamma \in \pi_1(\mathbb{P}^*, q(x))$, *and any unramified or tangential base point* $x \in D$.

Proof. Parts (a) and (b) follow from the local picture. For (c), note that $\delta x_0 \varphi_{\delta x_0}(\gamma) = \gamma\delta x_0 = \delta\delta^{-1}\gamma\delta x_0$, which forces $x_0\varphi_{\delta x_0}(\gamma) = \delta^{-1}\gamma\delta x_0$. This precisely says that $\varphi_{x_0}(\delta^{-1}\gamma\delta) = \varphi_{\delta x_0}(\gamma)$ as desired. The proof of (d) and (e) are similar. □

In the remainder of this section, we will show that $\Xi_\nu : \mathcal{C}_E^{pc} \to \mathcal{C}^{\succ}_{\mathbb{P}^1}$ is an equivalence.

LEMMA 4.4.4. *Given an object $(\pi : D \to \mathbb{P}^1, \alpha)$ of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$, suppose $\pi^{-1}(0)$ has cardinality n. Let*

$$\beta : \underbrace{\sqcup_{x \in \pi^{-1}(\{0,\infty\})} \mathrm{Spec}\, k}_{Z} \longrightarrow \underbrace{\sqcup_{i=1}^n \mathrm{Spec}\, k}_{W}$$

be the map identifying pairs of points according to α. Then the pushout $D_\alpha := D \cup_{Z,\beta} W$ exists in the category of schemes. We say that D_α is obtained by gluing D along α. Let $\nu : D \to D_\alpha$ be the canonical map. Then we have

(a) *The map $\nu : D \to D_\alpha$ is finite and surjective.*
(b) *ν identifies the topological space of D_α with the topological quotient of D by the equivalence relation given by α.*
(c) *For an open $U \subset D_\alpha$, we have $\Gamma(U, \mathcal{O}_{D_\alpha}) = \{f \in \Gamma(\nu^{-1}(U), \mathcal{O}_D) \mid f(x) = f(\alpha(x))$ for all $x \in \pi^{-1}(0)\}$.*
(d) *D_α is a prestable curve, and every point in $\nu(\pi^{-1}(\{0,\infty\}))$ is a node.*
(e) *ν identifies D with a normalization of D_α.*
(f) *ν restricts to an isomorphism on $D - \pi^{-1}(\{0,\infty\})$.*

(g) *Let (E_0, O) be the nodal elliptic curve obtained by gluing $\mathbb{P}^1$ along $\{0,\infty\}$, where we set the origin O to be the image of $1 \in \mathbb{P}^1$. Then the canonical map $D_\alpha \to E_0$ is precuspidal G-cover of E_0, and sending $(\pi : D \to \mathbb{P}^1, \alpha) \mapsto (\pi_\alpha : D_\alpha \to E_0)$ defines a "gluing" functor*

$$\mathrm{Glue} : \mathcal{C}_{\mathbb{P}^1}^{\succ} \longrightarrow \mathcal{C}_{E_0}^{pc}.$$

We say that D_α is the prestable curve obtained by gluing D along α. Note that if π is the degree 1 cover $D = \mathbb{P}^1 \to \mathbb{P}^1$, then D_α is a nodal cubic.

Proof. By [Stacks, 0E25], the pushout $D \cup_{Z,\beta} W$ exists and satisfies (b) and (c) (which in turn determines the pushout uniquely), and moreover the pushout diagram is also cartesian, so ν is the pullback of β and hence is finite surjective. Part (d) is [Knu83, Th. 3.4]. Since D is smooth, part (e) follows from the universal properties of normalization [Stacks, 035Q], and (f) follows from (e).

Finally, for (g), the G-equivariance of α implies that the G-action descends to D_α. Write $Z = \operatorname{Spec} R$ and $W = \operatorname{Spec} S$. Give W the unique G-action making $\beta : Z \to W$ G-equivariant. Note that $E_0 = \mathbb{P}^1 \cup_{Z/G} (W/G)$. The behavior of Glue on morphisms is defined by the universal property of pushouts, so it remains to show that $\pi_\alpha : D_\alpha \to E_0$ is a precuspidal G-cover. By (f), this is obvious on the smooth locus, so let $U = \operatorname{Spec} A \subset D$ be a G-invariant open affine containing $\pi^{-1}(\{0,\infty\})$ (see Lemma 2.4.1). Let $U_\alpha := \nu(U)$. Then $U_\alpha = \operatorname{Spec} A \times_R S$ and $U_\alpha/G = \operatorname{Spec}(A \times_R S)^G$. Since taking G-invariants is left exact, this is equal to $\operatorname{Spec} A^G \times_{R^G} S^G$, which is precisely the corresponding open affine neighborhood of the node in E_0, so π_α induces an isomorphism $D_\alpha/G \cong E_0$. This shows that π_α is finite. Since the maps $D \to D/G$, $Z \to Z/G$, $W \to W/G$ are all flat, so is $\pi_\alpha : D_\alpha \to E_0$ [Stacks, 0ECL]. Thus π_α is a G-cover. Part (f) implies that π_α sends nodes to nodes and is étale at smooth points not mapping to $O \in E_0$, so π_α is a precuspidal G-cover of E_0, as desired. ☐

THEOREM 4.4.5. *Let E be a nodal elliptic curve, and let $\nu : \mathbb{P}^1 \to E$ be a standard normalization. The categories $\mathcal{C}_E^{pc}, \mathcal{C}_{\mathbb{P}^1}^{\succ}$ are groupoids, and the functor*

$$\Xi_\nu : \mathcal{C}_E^{pc} \longrightarrow \mathcal{C}_{\mathbb{P}^1}^{\succ}$$

sending $p : C \to E$ to $(p' : C' \to \mathbb{P}^1, \alpha_p)$ is an equivalence of categories. We note that if ν' is another standard normalization (equivalently, $\nu' = \nu \circ \iota$), then ι defines an isomorphism $\Xi_\nu \cong \Xi_{\nu'}$. Let E_0 be the nodal elliptic curve obtained by gluing 0 to ∞ in $\mathbb{P}^1$. A quasi-inverse to Ξ_ν is given by composing the gluing functor of Lemma 4.4.4(g) with the isomorphism $\mathcal{C}_{E_0}^{pc} \xrightarrow{\sim} \mathcal{C}_E^{pc}$ induced by any isomorphism $E_0 \xrightarrow{\sim} E$.

Proof. By the universal properties of normalization [Stacks, 035Q], this normalization procedure defines a functor. Now consider a pair of objects

$(q_1 : D_1 \to \mathbb{P}^1, \alpha_1), (q_2 : D_2 \to \mathbb{P}^1, \alpha_2)$ in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Suppose we have a morphism $s : (q_1, \alpha_1) \to (q_2, \alpha_2)$ in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Since it respects α_1, α_2, by the universal property of pushouts, it induces a unique map $s_\alpha : (D_1)_{\alpha_1} \to (D_2)_{\alpha_2}$ in $\underline{\mathbf{Sch}}/\mathbb{P}^1$ fitting into a commutative diagram

$$\begin{tikzcd} D_1 \arrow[r, "s"] \arrow[d, "\nu_1"] & D_2 \arrow[d, "\nu_2"] \\ (D_1)_{\alpha_1} \arrow[r, "s_\alpha"] & (D_2)_{\alpha_2} \end{tikzcd} \tag{4.3}$$

of schemes over $\mathbb{P}^1$. By Lemma 4.4.4(e), the canonical maps ν_1, ν_2 are also normalization maps. The universal property of pushouts also gives unique maps $p_i : (D_i)_{\alpha_i} \to E_0$ making the diagrams

$$\begin{tikzcd} D_i \arrow[r, "\nu_i"] \arrow[d, "q_i"] & (D_i)_{\alpha_i} \arrow[d, "p_i"] \\ \mathbb{P}^1 \arrow[r] & E_0 \end{tikzcd}$$

commute (for $i = 1, 2$), and forming a commutative prism with (4.3). By Lemma 4.4.4(e) and (g), each p_i is a precuspidal G-cover and $q_i : D_i \to \mathbb{P}^1$ is a normalization of p_i. Pulling back p_i via some isomorphism $E \xrightarrow{\sim} E_0$, we find q_i is the normalization of a precuspidal G-cover of E, so Ξ_ν is essentially surjective.

Since ν_1, ν_2 are birational morphisms between separated schemes, s determines s_α, so Ξ_ν is faithful. Since s can be recovered from s_α as the induced map on normalizations Ξ_ν is fully faithful, so Ξ_ν is an equivalence.

Finally, to see that they are both groupoids, let $\mathrm{FEt}^G_{\mathbb{P}^*}$ denote the category of finite étale G-covers of $\mathbb{P}^*$. Consider the "forgetful functor" $\mathcal{C}^{\succ}_{\mathbb{P}^1} \to \mathrm{FEt}^G_{\mathbb{P}^*}$ sending (q, α) in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ to the restriction of q to $\mathbb{P}^*$. Since all schemes considered are separated, this functor is faithful. The category $\mathrm{FEt}^G_{\mathbb{P}^*}$ is a groupoid. (Any G-equivariant map between étale G-covers is an isomorphism.) Since the inverse of a map which preserves the identifications of fibers (in $\mathcal{C}^{\succ}_{\mathbb{P}^1}$) must also preserve identifications of fibers, this functor is also conservative (i.e., "isomorphism reflecting"), and hence $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ is a groupoid, and hence so is $\mathcal{C}^{pc}_E$. $\square$

Definition 4.4.6. We say that an object $(C' \to \mathbb{P}^1, \alpha)$ of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$ is *connected* if the precuspidal G-curve C'_α is connected.

4.5. *Balanced objects of* $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Here we describe the property of being balanced in terms of the internal logic of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$.

Let $(q : D \to \mathbb{P}^1, \alpha)$ be an object of $\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Let $x_0 \in D_{t_0}$ be a point, and let $\delta : t_0 \rightsquigarrow t_\infty$ be a path. Then $\alpha([x_0]) = [\delta x_0 \cdot h]$ for some $h \in G$, where h is unique up to the coset

$$H_{\delta, x_0} := G_{[\delta x_0]} h \in G_{[\delta x_0]} \backslash G.$$

Thus the coset $H_{\delta, x_0} \subset G$ is a well-defined function of the quadruple (q, α, δ, x_0).

LEMMA 4.5.1. *Let δ' be another path from $t_0 \rightsquigarrow t_\infty$. Then for any $h \in H_{\delta,x_0}$, we have*

$$[\delta' x_0 \cdot \varphi_{x_0}(\delta'^{-1}\delta)h] = [\delta x_0 \cdot h], \quad \textit{or equivalently} \quad \varphi_{x_0}(\delta'^{-1}\delta)h \in H_{\delta',x_0}.$$

Proof. The first equality follows from the definition of φ_{x_0}, and the equivalence follows from the definition of H_{δ,x_0} and H_{δ',x_0}. $\square$

The following proposition describes the objects of $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ which correspond to balanced objects of $\mathcal{C}_E^{pc}$.

PROPOSITION 4.5.2. *For an object $(q : D \to \mathbb{P}^1, \alpha)$ in $\mathcal{C}_{\mathbb{P}^1}^{\succ}$, the following are equivalent*:

(a) *For some (equivalently any) choices of $x_0 \in D_{t_0}$ and $x_\infty \in D_{t_\infty}$ satisfying $[x_\infty] = \alpha([x_0])$, we have*

$$\varphi_{x_0}(\gamma_0)^{-1} = \varphi_{x_\infty}(\gamma_\infty).$$

(b) *For some (equivalently any) choices of $x_0 \in D_{t_0}$, $\delta : t_0 \rightsquigarrow t_\infty$, and $h \in H_{\delta,x_0}$, we have*

$$\varphi_{x_0}(\gamma_0)^{-1} = h^{-1}\varphi_{x_0}(\gamma_\infty^\delta)h. \tag{4.4}$$

Moreover, if $p : C \to E$ is a precuspidal G-cover, then p is balanced if and only if $\Xi(p) = (p', \alpha_p) \in \mathcal{C}_{\mathbb{P}^1}^{\succ}$ satisfies either of the equivalent conditions (a) or (b).

Definition 4.5.3. We say that an object (q, α) of $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ is balanced if the equivalent conditions (a), (b) of Proposition 4.5.2 are satisfied.

Proof of Proposition 4.5.2. Fix a standard normalization $\nu : \mathbb{P}^1 \to E$. Since $\Xi_\nu : \mathcal{C}_E^{pc} \to \mathcal{C}_{\mathbb{P}^1}^{\succ}$ is an equivalence, we may assume that $q : D \to \mathbb{P}^1$ fits into a commutative diagram

$$\begin{tikzcd} D \arrow[r, "\nu_C"] \arrow[d, "q"] & C \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, "\nu"] & E \end{tikzcd}$$

such that ν_C, ν are normalization maps, q is the map of normalizations induced by p, and α is the G-equivariant bijection $D_0 \xrightarrow{\sim} D_\infty$ induced by ν_C. Equivalently, $(q, \alpha) \cong \Xi(p)$. We wish to show that p is balanced if and only if (q, α) satisfies (a) if and only if it satisfies (b).

By Proposition 4.4.3(a), $\varphi_{x_0}(\gamma_0)$ generates $G_{[x_0]}$ and is the unique element of $G_{[x_0]}$ inducing ζ_n on the cotangent space at $[x_0]$. Let $x_\infty \in D_{t_\infty}$ satisfying $[x_\infty] = \alpha([x_0])$. Again by Proposition 4.4.3(a), $\varphi_{x_\infty}(\gamma_\infty)$ is the unique element of $G_{[x_\infty]} = G_{\alpha([x_0])}$ inducing ζ_n on the cotangent space at $[x_\infty]$. Since $G_{[x_0]} = G_{\alpha_p([x_0])} = G_{[x_\infty]}$ (due to G-equivariance of α), we find that p is balanced at $\nu_C([x_0]) = \nu_C([x_\infty])$ if and only if

$$\varphi_{x_0}(\gamma_0)^{-1} = \varphi_{x_\infty}(\gamma_\infty). \tag{4.5}$$

In this case, using Proposition 4.4.3(e), for any $g \in G$ we have

$$\varphi_{x_0 g}(\gamma_0)^{-1} = g^{-1}\varphi_{x_0}(\gamma_0)^{-1}g = g^{-1}\varphi_{x_\infty}(\gamma_\infty)g = \varphi_{x_\infty g}(\gamma_\infty).$$

Since $\alpha([x_0 g]) = \alpha([x_0])g = [x_\infty]g = [x_\infty g]$, we have shown that p is balanced if and only if it is balanced at the image of x_0 if and only if (4.5) is satisfied. Thus, the choices of x_0, x_∞ are irrelevant, so in part (a), "some" is equivalent to "any," and p being balanced is equivalent to (a).

For part (b), we note that having fixed x_0, x_∞ satisfies $[x_\infty] = \alpha([x_0])$ if and only if $x_\infty = \delta x_0 h$ for some $\delta : t_0 \rightsquigarrow t_\infty$ and $h \in H_{\delta, x_0}$. Let x_∞ be given in this way. Then using Proposition 4.4.3, we find that

$$h^{-1}\varphi_{x_0}(\gamma_\infty^\delta)h = h^{-1}\varphi_{\delta x_0}(\gamma_\infty)h = \varphi_{\delta x_0 h}(\gamma_\infty) = \varphi_{x_\infty}(\gamma_\infty).$$

This implies that (b) is equivalent to (a). □

4.6. *Galois correspondence for precuspidal G-covers*. Recall that t_0 denotes a tangential base point at $0 \in \mathbb{P}^1$, and $t_\infty := \iota(t_0)$ is the corresponding tangential base point at $\infty \in \mathbb{P}^1$. In the remainder of Section 4, let $\Pi := \pi_1(\mathbb{P}^*, t_0)$, and let δ be a *good path* $t_0 \rightsquigarrow t_\infty$ (cf. Definition 4.3.1). Then Π is a free profinite group of rank 2 topologically generated by $\gamma_0, \gamma_\infty^\delta := \delta^{-1}\gamma\delta$.

Definition 4.6.1. For a good path $\delta : t_0 \rightsquigarrow t_\infty$, let $\underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ denote the category of pairs (F, α) where F is a finite set equipped with a left Π-action which commutes with a free and transitive right G-action, and

$$\alpha : \langle\gamma_0\rangle\backslash F \xrightarrow{\sim} \langle\gamma_\infty^\delta\rangle\backslash F$$

is a G-equivariant bijection (which we view as a "combinatorial gluing datum"). We will call such a pair (F, α) a *precuspidal G-datum* (relative to δ). Morphisms are given by (Π, G)-equivariant maps respecting α's. Let $F_\delta^\succ$ be the functor

$$F_\delta^\succ : \mathcal{C}_{\mathbb{P}^1}^\succ \longrightarrow \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$$

which sends $(q : D \to \mathbb{P}^1, \alpha)$ to the pair $(D_{t_0}, F_\delta^\succ(\alpha))$, where $F_\delta^\succ(\alpha)$ is the bijection

$$F_\delta^\succ(\alpha) \ : \ \langle\gamma_0\rangle\backslash D_{t_0} \xrightarrow{\xi_0} D_0 \xrightarrow{\alpha} D_\infty \xrightarrow{\xi_\infty^{-1}} \langle\gamma_\infty\rangle\backslash D_{t_\infty} \xrightarrow{\delta^{-1}} \langle\gamma_\infty^\delta\rangle\backslash D_{t_0},$$

where ξ_0, ξ_∞ are bijections defined as in Proposition 4.2.3(b).

Informally, an object (F, α) represents a three-point-cover $X \to \mathbb{P}^1$ with gluing data given by α, whose fibers at t_0 and t_∞ are both represented by F, and where the bijection $\delta : X_{t_0} \xrightarrow{\sim} X_{t_\infty}$ between the fibers becomes "normalized" to be the identity $\mathrm{id}_F : F \xrightarrow{\sim} F$. The orbit spaces $\langle\gamma_0\rangle\backslash F$ and $\langle\gamma_\infty^\delta\rangle\backslash F$ correspond to the fibers X_0, X_∞, and α corresponds to the gluing data $X_0 \xrightarrow{\sim} X_\infty$.

PROPOSITION 4.6.2. *The functor $F_\delta^\succ : \mathcal{C}_{\mathbb{P}^1}^\succ \longrightarrow \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ is an equivalence of categories.*

Proof. Given objects $(D,\alpha),(D',\alpha') \in \mathcal{C}_{\mathbb{P}^1}^{\succ}$, the usual Galois correspondence implies giving a G-equivariant morphisms $f : D \to D'$ over $\mathbb{P}^1$ is the same as giving a (Π, G)-equivariant map $D_{t_0} \to D'_{t_0}$. We must show that f respects α if and only if the induced map $f_* : \langle\gamma_0\rangle\backslash D_{t_0} \to \langle\gamma_0\rangle\backslash D'_{t_0}$ respects $F_\delta^{\succ}(\alpha), F_\delta^{\succ}(\alpha')$. Consider the diagram

$$\begin{array}{ccccccccc}
\langle\gamma_0\rangle\backslash D_{t_0} & \xrightarrow{\xi_0} & D_0 & \xrightarrow{\alpha} & D_\infty & \xrightarrow{\xi_\infty^{-1}} & \langle\gamma_\infty\rangle\backslash D_{t_\infty} & \xrightarrow{\delta^{-1}} & \langle\gamma_\infty^\delta\rangle\backslash D_{t_0} \\
\big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\
\langle\gamma_0\rangle\backslash D'_{t_0} & \xrightarrow{\xi_0} & D'_0 & \xrightarrow{\alpha} & D'_\infty & \xrightarrow{\xi_\infty^{-1}} & \langle\gamma_\infty\rangle\backslash D'_{t_\infty} & \xrightarrow{\delta^{-1}} & \langle\gamma_\infty^\delta\rangle\backslash D'_{t_0}.
\end{array}$$

The composition of the top row is just $F_\delta^{\succ}(\alpha)$, and the composition of the bottom row is $F_\delta^{\succ}(\alpha')$. From left to right, the first, third, and fourth squares commute because $\xi_0, \xi_\infty^{-1}, \delta^{-1}$ are all functorial in D. Thus the diagram commutes if and only if the second square commutes, which happens if and only if f respects α. Thus $F_\delta^{\succ}$ is fully faithful.

To show that $F_\delta^{\succ}$ is essentially surjective, we will define a quasi-inverse functor, denoted $H : \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ} \to \mathcal{C}_{\mathbb{P}^1}^{\succ}$. Given a precuspidal G-datum (F,α), by the usual Galois correspondence we obtain a G-cover $\pi : X \to \mathbb{P}^1$, étale over $\mathbb{P}^*$, equipped with a (Π,G)-equivariant bijection $\varphi : F \xrightarrow{\sim} X_{t_0}$. Moreover, δ defines a G-equivariant bijection $\delta : X_{t_0} \xrightarrow{\sim} X_{t_\infty}$, which induces a G-equivariant bijection $\langle\gamma_\infty^\delta\rangle\backslash X_{t_0} \xrightarrow{\sim} \langle\gamma_\infty\rangle\backslash X_{t_\infty}$. Using $\varphi : F \xrightarrow{\sim} X_{t_0}$ and $\delta\circ\varphi : F \xrightarrow{\sim} X_{t_\infty}$, we obtain G-equivariant bijections

$$\begin{array}{ccccc}
\langle\gamma_0\rangle\backslash F & \xrightarrow{\varphi} & \langle\gamma_0\rangle\backslash X_{t_0} & \xrightarrow{\xi_0} & X_0, \\
\langle\gamma_\infty^\delta\rangle\backslash F & \xrightarrow{\delta\circ\varphi} & \langle\gamma_\infty\rangle\backslash X_{t_\infty} & \xrightarrow{\xi_\infty} & X_\infty.
\end{array}$$

Connecting these bijections via α, we obtain a G-equivariant bijection $\alpha_\pi : X_0 \xrightarrow{\sim} X_\infty$, and we define $H(F,\alpha) = (\pi : X \to \mathbb{P}^1, \alpha_\pi)$. It is straightforward to check that $F_\delta^{\succ} \circ H \cong \mathrm{id}_{\underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}}$, so $F_\delta^{\succ}$ is essentially surjective. $\square$

4.7. *Combinatorial balance*, *combinatorial connectedness*. For a good path $\delta : t_0 \rightsquigarrow t_\infty$, we have defined equivalences of categories

$$\mathcal{C}_E^{pc} \xrightarrow{\Xi} \mathcal{C}_{\mathbb{P}^1}^{\succ} \xrightarrow{F_\delta^{\succ}} \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}.$$

Next we will describe when a precuspidal G-datum corresponds to a balanced object of $\mathcal{C}_{\mathbb{P}^1}^{\succ}$. Given a precuspidal G-datum (F,α) (relative to a good path $\delta : t_0 \rightsquigarrow t_\infty$), let $[\cdot]_0, [\cdot]_\infty$ be the (G-equivariant) projections

$$[\cdot]_0 : F \longrightarrow \langle\gamma_0\rangle\backslash F,$$
$$[\cdot]_\infty : F \longrightarrow \langle\gamma_\infty^\delta\rangle\backslash F.$$

For $x \in F$, we have a homomorphism

$$\varphi_x : \Pi \longrightarrow G \qquad \text{defined by} \quad \gamma\cdot x = x\cdot\varphi_x(\gamma) \quad \text{for all } \gamma\in\Pi.$$

Note that we have $G_{[x]_0} = \varphi_x(\langle\gamma_0\rangle)$ and $G_{[x]_\infty} = \varphi_x(\langle\gamma_\infty^\delta\rangle)$. Finally for $x \in F$, let

$$H_{\delta,x} := \{h \in G \mid \alpha([x]_0) = [x]_\infty \cdot h\} \in G_{[x]_\infty}\backslash G. \tag{4.6}$$

Definition 4.7.1. We say a precuspidal G-datum $(F,\alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ is *balanced* if for some (equivalently any) choices of $x \in F$ and $h \in H_{\delta,x}$, we have

$$\varphi_x(\gamma_0)^{-1} = h^{-1}\varphi_x(\gamma_\infty^\delta)h.$$

It follows from Proposition 4.5.2 that this definition makes sense and agrees with the notion of balanced objects in $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ relative to the equivalence $F_\delta^{\succ}$.

Next we want to express the notion of connectedness for precuspidal G-covers in terms of precuspidal G-data. For this it is useful to introduce the graph of components of a prestable curve.

Definition 4.7.2. A graph Γ consists of the data $(\mathcal{V}_\Gamma, \mathcal{E}_\Gamma, \sigma, \tau)$ where

- $\mathcal{V}_\Gamma$ is a set, called the set of *vertices* of Γ;
- $\mathcal{E}_\Gamma$ is a set called the set of (directed) *edges* (or arrows) of Γ;
- $\sigma : \mathcal{E}_\Gamma \to \mathcal{V}_\Gamma$ is a function, called the "incidence map" or the "source map;"
- $\tau : \mathcal{E}_\Gamma \to \mathcal{E}_\Gamma$ is a fixed-point free involution. For $e \in \mathcal{E}_\Gamma$, we also write $\overline{e} := \tau(e)$.

If $e \in \mathcal{E}_\Gamma$, then we say that e is an arrow from $\sigma(e)$ to $\sigma(\overline{e})$, and we write $\sigma(e) \xrightarrow{e} \sigma(\overline{e})$.

A morphism of graphs is given by a pair of functions between the vertex and edge sets which commute with the σ, τ maps. Let $\underline{\mathbf{Graphs}}$ denote the category of graphs. See [Bas93].

Let $\pi : C \to E$ be a precuspidal G-torsor. Let $(\pi' : C' \to \mathbb{P}^1, \alpha_\pi : C'_0 \xrightarrow{\sim} C'_\infty) = \Xi(\pi)$ be the associated object of $\mathcal{C}_{\mathbb{P}^1}^{\succ}$ obtained by normalization. Its component graph Γ_π is given by

- $\mathcal{V}_{\Gamma_\pi} = \pi_0(C')$ is the set of components of C';
- $\mathcal{E}_{\Gamma_\pi} = C'_0 \sqcup C'_\infty$;
- for $e \in C'_0 \sqcup C'_\infty$, $\sigma(e)$ is the component on which e lies;
- for $e \in C'_0$, $\overline{e} = \alpha_\pi(e)$, and if $e \in C'_\infty$, then $\overline{e} := \alpha_\pi^{-1}(e)$.

The action of G on C induces an action on Γ_π. It is immediate that

Proposition 4.7.3. *C is connected if and only if Γ_π is connected.*

Let $\delta : t_0 \rightsquigarrow t_\infty$ be a good path in $\mathbb{P}^*$, and let $(F,\alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ be an object. Let $\Gamma_{F,\alpha}$ denote the graph given by

- $\mathcal{V}_{\Gamma_{F,\alpha}} = \Pi\backslash F$;
- $\mathcal{E}_{\Gamma_{F,\alpha}} = \langle\gamma_0\rangle\backslash F \sqcup \langle\gamma_\infty^\delta\rangle\backslash F$;
- for $e \in \mathcal{E}_{\Gamma_{F,\alpha}}$, either $e = [x]_0$ or $e = [x]_\infty$ for some $x \in F$ — in either case, let $\sigma(e) := \Pi \cdot x$;

- for $e \in \langle\gamma_0\rangle\backslash F$, define $\overline{e} := \alpha(e)$, and for $e \in \langle\gamma_\infty^\delta\rangle\backslash F$, define $\overline{e} := \alpha^{-1}(e)$.

Note that $\Gamma_{F,\alpha}$ comes with a natural action of G induced by its action on F.

PROPOSITION 4.7.4. *The graph associated to a precuspidal G-cover $\pi : C \to E$ is isomorphic to the graph associated to the precuspidal G-datum $F_\delta^\succ(\Xi(\pi))$. Given a precuspidal G-datum (F,α) and $x \in F$, let $H_{\delta,x}$ be as in* (4.6). *Let $M_x := \varphi_x(\Pi)$ be the "monodromy group at x." Then the graph $\Gamma_{F,\alpha}$ is connected if and only if either of the two equivalent conditions hold*:

(a) *For some* (*equivalently any*) $x \in F$, *G is generated by M_x and $H_{\delta,x}$.*
(b) *For some* (*equivalently any*) $x \in F$, *G is generated by M_x and any element $h \in H_{\delta,x}$.*

Proof. That the graphs of π and $F_\delta^\succ(\Xi(\pi))$ are isomorphic follows from their definitions. Since $G_{[x]_\infty} = \varphi_x(\langle\gamma_\infty^\delta\rangle) \subset M_x$, we see that (a) and (b) are equivalent. It remains to show that they are equivalent to the connectedness of $\Gamma_{F,\alpha}$. Suppose $\Gamma_{F,\alpha}$ is connected. Then for any $g \in G$, there is a path in $\Gamma_{F,\alpha}$ from the vertex Πx to Πxg. Setting $g_n = g, g_0 = 1$, this means there is a sequence $1 = g_1, g_2, \ldots, g_n = g \in G$ and edges $e_1, e_2, \ldots, e_n$ fitting into a path

$$\Pi x = \Pi x g_0 \xrightarrow{e_1} \Pi x g_1 \xrightarrow{e_2} \Pi x g_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} \Pi x g_n = \Pi x g. \tag{4.7}$$

For $i \in [1,n]$, there are two possibilities for the edge $\Pi x g_{i-1} \xrightarrow{e_i} \Pi x g_i$:

- $e_i = [\gamma x g_{i-1}]_0 = [x\varphi_x(\gamma)g_{i-1}]_0$ for some $\gamma \in \Pi$. In this case, since $\alpha([x]_0) = [x \cdot h]_\infty$ for any $h \in H_{\delta,x}$, by G-equivariance we have $\overline{e_i} = [xh\varphi_x(\gamma)g_{i-1}]_\infty$. The fact that $\sigma(\overline{e_i}) = \Pi x g_i$ says precisely that
$$[xh\varphi_x(\gamma)g_{i-1}]_\infty = \langle\gamma_\infty^\delta\rangle xh\varphi_x(\gamma)g_{i-1} \subset \Pi x g_i.$$
Thus, for some $\gamma' \in \Pi$, we have
$$xh\varphi_x(\gamma)g_{i-1} = \gamma' x g_i = x\varphi_x(\gamma')g_i \qquad \text{so} \qquad g_i \in M_x h M_x g_{i-1}.$$
- $e_i = [\gamma x g_{i-1}]_\infty = [x\varphi_x(\gamma)g_{i-1}]_\infty$ for some $\gamma \in \Pi$. In this case, a similar argument shows $\overline{e_i} = [xh^{-1}\varphi_x(\gamma)g_{i-1}]_0$. The fact that $\sigma(\overline{e_i}) = \Pi x g_i$ says precisely that $[xh^{-1}\varphi_x(\gamma)g_{i-1}]_0 = \langle\gamma_\infty^\delta\rangle xh^{-1}\varphi_x(\gamma)g_{i-1} \subset \Pi x g_i$. Thus, for some $\gamma' \in \Pi$, we have
$$xh^{-1}\varphi_x(\gamma)g_{i-1} = \gamma' x g_i = x\varphi_x(\gamma')g_i \qquad \text{so} \qquad g_i \in M_x h^{-1} M_x g_{i-1}.$$

Thus, by induction we find that for $j = \{1, \ldots, n\}$, there exist $i_j \in \{\pm 1\}$ and $m_j, m_j' \in M_x$ such that
$$g_n = g = m_n' h^{i_n} m_n m_{n-1}' h^{i_{n-1}} m_{n-1} \cdots m_1' h^{i_1} m_1.$$
Since this holds for any $g \in G$, we find that G is generated by M_x and h.

Conversely, suppose G is generated by M_x and h for some $h \in H_{\delta,x}$. Then for every $g \in G$, we may write it as $g = h^{i_n} m_n h^{i_{n-1}} m_{n-1} \cdots h^{i_1} m_1$ with

$i_j \in \{\pm 1\}$ and $m_j \in M_x$. Then for $k \in [0,n]$, define g_k inductively by $g_0 = 1$ and $g_k = h^{i_k} m_k g_{k-1}$ for $k \geq 1$ (so $g_n = g$). Define edges $e_1 \ldots, e_n$ by

$$e_k := \begin{cases} [xm_kg_{k-1}]_0 & \text{if } i_j = 1, \\ [xm_kg_{k-1}]_\infty & \text{if } i_j = -1. \end{cases}$$

Then $e_1, \ldots, e_n$ define a path

$$\Pi x = \Pi x g_0 \xrightarrow{e_1} \Pi x g_1 \xrightarrow{e_2} \Pi x g_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} \Pi x g_n = \Pi x g.$$

Since this holds for every $g \in G$, this implies that $\Gamma_{F,\alpha}$ is connected. Since x was arbitrary in the discussion above, it follows that "some" is equivalent to "any" in (a) and (b). $\square$

Definition 4.7.5. For a good path $\delta : t_0 \rightsquigarrow t_\infty$, we say that a precuspidal G-datum $(F, \alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ is *connected* if the equivalent conditions of Proposition 4.7.4 hold. A precuspidal G-datum is *cuspidal* if it is connected and balanced.

Proposition 4.7.6. *For a good path $\delta : t_0 \rightsquigarrow t_\infty$, the equivalences $\Xi : \mathcal{C}_E^{pc} \xrightarrow{\sim} \mathcal{C}_{\mathbb{P}^1}^{\succ}$ and $F_\delta^{\succ} : \mathcal{C}_{\mathbb{P}^1}^{\succ} \xrightarrow{\sim} \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ restrict to equivalences between the full subcategories of connected objects, balanced objects, and connected+balanced objects.*

Proof. This follows from the definitions together with Propositions 4.5.2 and 4.7.4. $\square$

4.8. *Parametrizing isomorphism classes of cuspidal admissible G-covers: the δ-invariant.* As usual, let $\Pi := \pi_1(\mathbb{P}^*, t_0)$, and let $\delta : t_0 \rightsquigarrow t_\infty$ be a good path in $\mathbb{P}^*$.

Proposition 4.8.1. *A precuspidal G-datum $(F, \alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ is cuspidal (i.e., balanced and connected) if and only if both of the following conditions hold:*

(a) *For some (equivalently any) $x \in F$ and $h \in H_{\delta,x}$ (see (4.6)), we have*

$$\varphi_x(\gamma_0)^{-1} = h^{-1}\varphi_x(\gamma_\infty^\delta)h \qquad \text{or equivalently} \qquad \varphi_x(\gamma_\infty^\delta) = h\varphi_x(\gamma_0)^{-1}h^{-1}.$$

(b) *For some (equivalently any) $x \in F$ and $h \in H_{\delta,x}$, G is generated by h and $\varphi_x(\gamma_0)$.*

Proof. Condition (a) is the definition of balancedness. The group Π is generated by γ_0 and γ_∞^δ, and hence in the presence of (a), G is generated by $M_x := \varphi_x(\Pi)$ and h if and only if it is generated by $\varphi_x(\gamma_0)$ and h. Thus in the presence of (a), by Proposition 4.7.4, (b) is equivalent to (F, α) being connected. $\square$

We will define a bijection between the set of isomorphism classes of cuspidal objects of $\underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ and an explicit finite set built from G. Let G act on the set $G \times G$ by conjugation:

$$g \cdot (u, h) = (gug^{-1}, ghg^{-1}).$$

If two pairs (u, h) and (u', h') are conjugate by this action, then we will write $(u, h) \sim (u', h')$. Let $\mathbb{Z}$ act on $G \times G$ on the right by the rule

$$(u, h) \cdot k = (u, u^k h), \qquad k \in \mathbb{Z}.$$

These actions of G and $\mathbb{Z}$ on $G \times G$ commute and they preserve the subset of generating pairs. Let $\mathbb{I}(G)$ denote the subset of the orbit space $G \backslash (G \times G)/\mathbb{Z}$ represented by generating pairs:

$$\mathbb{I}(G) := G \backslash \{(u, h) : u, h \text{ generate } G\}/\mathbb{Z}.$$

Definition 4.8.2. For a generating pair $(u, h) \in G \times G$, let $[\![u, h]\!]$ denote its image in $\mathbb{I}(G)$.

Definition 4.8.3. For a generating pair $(u, h) \in G \times G$, let $(F_{u,h}, \alpha_{u,h}) \in \underline{\mathbf{Sets}}^{(\Pi, G)_{\delta}, \succ}$ be given as follows:

- $F_{u,h}$ is the pointed set G (with distinguished element 1_G) viewed as a right G-torsor via right-multiplication. Give $F_{u,h}$ the structure of a left Π-set using the unique left Π-action which both commutes with the right G-action and also satisfies

$$\gamma_\infty^\delta \cdot 1_G = 1_G u \qquad \text{and} \qquad \gamma_0 \cdot 1_G = 1_G h^{-1} u^{-1} h.$$

  Explicitly, this left Π-action has the following description:

$$\gamma_\infty^\delta \cdot x = ux, \qquad \gamma_0 \cdot x = h^{-1} u^{-1} hx \qquad \forall x \in F_{u,h},$$

  where the right-hand sides of the equalities involve multiplication in G. Thus, the Π action is given by left multiplication by the subgroup

$$M_{u,h} := \langle u, h^{-1} u^{-1} h \rangle \le G = \varphi_{1_G}(\Pi). \tag{4.8}$$

  We call this subgroup the *monodromy group* at 1_G.
- $\alpha_{u,h}$ is the G-equivariant bijection

$$\begin{aligned} \alpha_{u,h} : \langle \gamma_0 \rangle \backslash F_{u,h} &\longrightarrow \langle \gamma_\infty^\delta \rangle \backslash F_{u,h}, \\ \langle \gamma_0 \rangle \cdot x &\mapsto \langle \gamma_\infty^\delta \rangle \cdot hx, \end{aligned}$$

  where $x \in F_{u,h}$ and hx is multiplication in G. We note that this is the unique G-equivariant bijection satisfying $\alpha_{u,h}([1]_0) = [1]_\infty \cdot h = [h]_\infty$. Equivalently, in terms of the Π-action, we have

$$\begin{aligned} \alpha_{u,h} : \langle h^{-1} u^{-1} h \rangle \backslash G &\longrightarrow \langle u \rangle \backslash G, \\ \langle h^{-1} u^{-1} h \rangle \cdot x &\mapsto \langle u \rangle \cdot hx. \end{aligned}$$

  Note that $H_{\delta, 1_G} = \varphi_{1_G}(\langle \gamma_\infty^\delta \rangle) \cdot h = \langle u \rangle h$.

It is easy to check that $(F_{u,h}, \alpha_{u,h})$ is balanced and connected, hence cuspidal.

THEOREM 4.8.4 (The δ-invariant). *Let $\delta : t_0 \rightsquigarrow t_\infty$ be a good path, and let $(F, \alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ be a precuspidal G-datum. For any $x \in F$ and $h \in H_{\delta,x}$, define*

$$\begin{aligned}\mathrm{Inv}_\delta : \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}} &\longrightarrow G\backslash(G\times G)/\mathbb{Z},\\ (F,\alpha) &\mapsto [\![\varphi_x(\gamma_\infty^\delta), h]\!].\end{aligned}$$

We will say that $\mathrm{Inv}_\delta(F,\alpha)$ is the invariant *of (F,α) with respect to the path δ. Let $\mathcal{D}_\delta$ be the full subcategory of $\underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ consisting of cuspidal objects, and let $\pi_0(\mathcal{D}_\delta)$ denote the set of isomorphism classes of $\mathcal{D}_\delta$. Then $\mathrm{Inv}_\delta(F,\alpha)$ is independent of x and h and its restriction to $\mathcal{D}_\delta$ yields a bijection*

$$\mathrm{Inv}_\delta : \pi_0(\mathcal{D}_\delta) \xrightarrow{\sim} \mathbb{I}(G)$$

with inverse induced by $(u,h) \mapsto (F_{u,h}, \alpha_{u,h})$.

Definition 4.8.5. Let E be a nodal elliptic curve. Given an object of $\mathcal{C}_E^{pc}$ or $\mathcal{C}_{\mathbb{P}^1}^{\succ}$, its δ-invariant is the δ-invariant of its image in $\underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ via the equivalences Ξ and $F_\delta^\succ$. In particular, using the equality $\mathcal{C}_E^c = \mathcal{A}dm(G)_E$, taking δ-invariants induces a bijection

$$\pi_0(\mathcal{A}dm(G)_E) \xrightarrow{\sim} \mathbb{I}(G).$$

Proof of Theorem 4.8.4. First we note that for $x' = xg \in F$, we have

$$\varphi_{xg}(\gamma) = g^{-1}\varphi_x(\gamma)g \quad \forall\gamma\in\Pi \quad \text{and} \quad H_{\delta,xg} = g^{-1}H_{\delta,x}g.$$

Thus, changing x results in a conjugate pair, so Inv_δ is independent of x. On the other hand, the only other choices of h are given by $\varphi_x(\gamma_\infty^\delta)^k \cdot h$, so by the definition of the $\mathbb{Z}$-action on $(G\times G)$, Inv_δ is also independent of h.

Next we note that for any choice of x, h, if (F,α) lies in $\mathcal{D}_\delta$, then by connectedness of (F,α), G is generated by $\varphi_x(\gamma_0), \varphi_x(\gamma_\infty^\delta)$, and h. On the other hand, since (F,α) is balanced, $\varphi_x(\gamma_0) = h^{-1}\varphi_x(\gamma_\infty^\delta)^{-1}h$, so G is also generated by $\varphi_x(\gamma_\infty^\delta), h$. Thus Inv_δ applied to an element of $\mathcal{D}_\delta$ represents an element of $\mathbb{I}(G)$.

Next, if $(F,\alpha) \cong (F',\alpha')$ via a (Π,G)-equivariant bijection $f : F \xrightarrow{\sim} F'$ respecting α, α', then one checks that

$$\varphi_{f(x)}(\gamma) = \varphi_x(\gamma) \quad \forall x\in F, \gamma\in\Pi \qquad \text{and} \qquad H_{\delta,x} = H_{\delta,f(x)} \quad \forall x \in F$$

so Inv_δ is an isomorphism invariant.

We claim that the map $\{(u,h) \in G\times G \mid u,h \text{ generate } G\} \to \pi_0(\mathcal{D}_\delta)$ defined by sending (u,h) to the isomorphism class of $(F_{u,h},\alpha_{u,h})$ gives an inverse to Inv_δ. Indeed, we have $\mathrm{Inv}_\delta(F_{u,h},\alpha_{u,h}) = [\![u,h]\!]$ by construction. On the other hand, if $(F,\alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ has $\mathrm{Inv}_\delta(F,\alpha) = [\![u,h]\!]$, then for some $x \in F$, we have $\varphi_x(\gamma_\infty^\delta) = u$ and $h \in H_{\delta,x}$. Then one can check that the unique G-equivariant map $F_{u,h} \xrightarrow{\sim} F$ sending $1_G \mapsto x$ defines an isomorphism $(F, \alpha) \xrightarrow{\sim} (F_{u,h}, \alpha_{u,h})$ as desired. □

PROPOSITION 4.8.6. *Let $\delta : t_0 \rightsquigarrow t_\infty$ be a good path. Let $\pi : C \to E$ be a cuspidal G-cover with δ-invariant $\mathrm{Inv}_\delta(\pi) = [\![u, h]\!]$. Then its Higman invariant is the conjugacy class of $[u^{-1}, h^{-1}] := u^{-1}h^{-1}uh \sim [u, h]$.*

Proof. Let e denote the common ramification indices of π above O. Let $\Xi(\pi) = (\pi' : C' \to \mathbb{P}^1, \alpha)$ be the normalization-with-gluing data. Since normalization induces an isomorphism on the smooth locus, every point in $\pi'^{-1}(1)$ is also ramified with index e, and the Higman invariant of π is the Higman invariant of π' at $1 \in \mathbb{P}^1$. We note that the conjugacy class of $[u, h]$ does not depend on the representative $(u, h) \in [\![u, h]\!]$. Thus from the definition of Inv_δ, we may assume that for some $x_0 \in C'_{t_0}$, we have

$$u = \varphi_{x_0}(\gamma_\infty^\delta) \qquad \text{and} \qquad h \in H_{\delta, x_0}.$$

For any $x \in \pi'^{-1}(1)$, let $\chi_x : G_x \to \mathrm{GL}(T_x^* C')$ be the local representation as in (4.2). Then the Higman invariant of π is precisely $\chi_x^{-1}(\zeta_e)$. Since δ is good, for some path $\epsilon : t_0 \rightsquigarrow t_1$, we have $\gamma_1^\epsilon = (\gamma_\infty^\delta)^{-1}\gamma_0^{-1}$. By Proposition 4.2.3, there is a canonical G-equivariant bijection

$$\xi_1 : \langle \gamma_1 \rangle \backslash C'_{t_1} \xrightarrow{\sim} C'_1.$$

Let $x_1 := \epsilon x_0 \in C'_{t_1}$, and let $x := \xi_1([x_1])$. Then from the local picture (also see Proposition 4.4.3), we have

$$\chi_x^{-1}(\zeta_e) = \varphi_{x_1}(\gamma_1) = \varphi_{x_0}(\gamma_1^\epsilon)$$

but by our choice of ϵ, $\varphi_{x_0}(\gamma_1^\epsilon) = \varphi_{x_0}(\gamma_\infty^\delta)^{-1}\varphi_{x_0}(\gamma_0)^{-1} = u^{-1}\varphi_{x_0}(\gamma_0)^{-1}$, and since the G-action is balanced, we have $\varphi_{x_0}(\gamma_0)^{-1} = h^{-1}\varphi_{x_0}(\gamma_\infty^\delta)h = h^{-1}uh$. Thus we find that the Higman invariant of π is the conjugacy class of

$$\chi_x^{-1}(\zeta_e) = [u^{-1}, h^{-1}] \sim [u, h]. \qquad \square$$

4.9. *The δ-invariant of the $[-1]$-pullback of a cuspidal G-cover.* The sole purpose of this section is to prove the following proposition. To simplify calculations, we will assume that δ is a good path which is moreover *symmetric* (cf. Definition 4.3.1).

PROPOSITION 4.9.1. *Let $\iota \in \mathrm{Aut}(\mathbb{P}^1)$ denote the unique automorphism fixing 1 and switching $0, \infty$. Let t_0 be a tangential base point at $0 \in \mathbb{P}^1$, and let $t_\infty := \iota(t_0)$. Let $\delta : t_0 \rightsquigarrow t_\infty$ be a* symmetric *good path. Let E be a nodal elliptic curve over k. Let $p : C \to E$ be a cuspidal G-cover. If $\mathrm{Inv}_\delta(p) = [\![u, h]\!]$, then $\mathrm{Inv}_\delta([-1]^* p) = [\![u^{-1}, h^{-1}]\!]$. In particular, if $[-1]$ denotes the unique non-trivial automorphism of E, then the following are equivalent:*

(a) *$[-1]$ lifts to an automorphism of C;*

(b) $[-1]^*p \cong p$ *as cuspidal* G*-covers;*
(c) $[\![u,h]\!] = [\![u^{-1},h^{-1}]\!]$;
(d) (u^{-1},h^{-1}) *is conjugate to* $(u,u^r h)$ *for some* $r>0$.

Proof. The equivalence of (a), (b), (c), and (d) follows from the statement that $\mathrm{Inv}_\delta([-1]^*p) = [\![u^{-1},h^{-1}]\!]$, so this is what we will prove.

Via Ξ, $p: C\to E$ corresponds to an object $(q: D\to\mathbb{P}^1,\alpha)\in\mathcal{C}^{\succ}_{\mathbb{P}^1}$. Let $\overline{D} := \iota^*D$, so we have a cartesian diagram

$$\begin{tikzcd} \overline{D} \arrow[r, "\tilde\iota"] \arrow[d, "\overline{q}"] & D \arrow[d, "q"] \\ \mathbb{P}^1 \arrow[r, "\iota"] & \mathbb{P}^1 \end{tikzcd}$$

with $\tilde\iota$ a G-equivariant isomorphism. Via the standard normalization $\mathbb{P}^1\to E$, $[-1]$ induces the automorphism ι, so it suffices to show that if $\mathrm{Inv}_\delta(q,\alpha) = [\![u,h]\!]$, then $\mathrm{Inv}_\delta(\iota^*q,\iota^*\alpha) = [\![u^{-1},h^{-1}]\!]$.

The map $\tilde\iota$ induces bijections $\tilde\iota:\overline{D}_0\xrightarrow{\sim}D_\infty$ and $\tilde\iota:\overline{D}_\infty\xrightarrow{\sim}D_0$. Thus $\alpha: D_0\xrightarrow{\sim}D_\infty$ induces a bijection

$$\overline{\alpha} := \iota^*\alpha : \overline{D}_0 \xrightarrow{\tilde\iota} D_\infty \xrightarrow{\alpha^{-1}} D_0 \xrightarrow{\tilde\iota^{-1}} \overline{D}_\infty.$$

Let $x_0\in D_{t_0}$ be a point, and as usual let

$$H_{\delta,x_0} := \{h\in G \mid \alpha([x_0]) = [\delta x_0 h]\}\in G_{[\delta x_0]}\backslash G.$$

Fix $h\in H_{\delta,x_0}$. Then we have

$$\mathrm{Inv}_\delta(q,\alpha) = [\![\varphi_{x_0}(\gamma_\infty^\delta),h]\!].$$

Since δ is symmetric, write $\iota_*\delta = \gamma_0^r\delta^{-1}\gamma_\infty^s$ for some $r,s\in\mathbb{Z}$. Let $x_\infty := \gamma_\infty^{-s}\delta x_0 h\in D_{t_\infty}$, so that $\alpha([x_0]) = [x_\infty]$. The canonical isomorphism $F_{t_0}\circ\iota^*\cong F_{\iota(t_0)} = F_{t_\infty}$ evaluated at $q: D\to\mathbb{P}^1$ is realized by the bijection

$$\tilde\iota:\overline{D}_{t_0}\xrightarrow{\sim}D_{t_\infty},$$

which is G-equivariant and moreover (by definition of $\iota_*:\pi_1(\mathbb{P}^*,t_0)\to\pi_1(\mathbb{P}^*,t_\infty)$ satisfies

$$\tilde\iota(\gamma z) = i_*\gamma\cdot\tilde\iota(z)\qquad\text{for all } z\in\overline{D}_{t_0}\text{ and }\gamma\in\pi_1(\mathbb{P}^*,t_0). \tag{4.9}$$

In particular, this implies that $\tilde\iota:\overline{D}_{t_0}\to D_{t_\infty}$ commutes with $[\cdot]$, and the same is true for $\tilde\iota:\overline{D}_{t_\infty}\to D_{t_0}$.

Let $\overline{x_\infty} := \tilde\iota^{-1}(x_\infty)\in\overline{D}_{t_0}$. Then by (4.9) and G-equivariance of $\tilde\iota$, for any $\gamma\in\pi_1(\mathbb{P}^*,t_0)$, we have

$$x_\infty\varphi_{\overline{x_\infty}}(\gamma) = \tilde\iota(\overline{x_\infty}\cdot\varphi_{\overline{x_\infty}}(\gamma)) = \tilde\iota(\gamma\cdot\overline{x_\infty}) = \iota_*\gamma\cdot x_\infty = x_\infty\varphi_{x_\infty}(\iota_*\gamma). \tag{4.10}$$

Since $\iota_*\delta = \gamma_0^r\delta^{-1}\gamma_\infty^s$, we have $\iota_*(\gamma_\infty^\delta) = \gamma_0^{\delta^{-1}\gamma_\infty^s}$. Thus since the G-action on D_{t_∞} is free, we get

$$\begin{aligned}
\varphi_{\overline{x_\infty}}(\gamma_\infty^\delta) &= \varphi_{x_\infty}(\gamma_0^{\delta^{-1}\gamma_\infty^s}) && \text{take } \gamma = \gamma_\infty^\delta \text{ in (4.10)},\\
&= \varphi_{\delta^{-1}\gamma_\infty^s x_\infty}(\gamma_0) && \text{by Proposition 4.4.3(d)},\\
&= \varphi_{x_0 h}(\gamma_0) && \text{since } x_\infty := \gamma_\infty^{-s}\delta x_0 h,\\
&= h^{-1}\varphi_{x_0}(\gamma_0)h && \text{by Proposition 4.4.3(e)},\\
&= h^{-2}\varphi_{x_0}(\gamma_\infty^\delta)^{-1}h^2 && \text{since } q \text{ is balanced (see Proposition 4.5.2)}.
\end{aligned}$$

computes the first part of $\mathrm{Inv}(\iota^*q, \iota^*\alpha)$. To compute the second part, note that

$$\begin{aligned}
\overline{\alpha}([\overline{x_\infty}]) &= \tilde{\iota}^{-1}(\alpha^{-1}([x_\infty]))\\
&= \tilde{\iota}^{-1}(\alpha^{-1}([\gamma_\infty^{-s}\delta x_0 h])) = \tilde{\iota}^{-1}(\alpha^{-1}([\delta x_0 h])) = \tilde{\iota}^{-1}([x_0]).
\end{aligned} \tag{4.11}$$

On the other hand, for $g \in G$ we have

$$\tilde{\iota}(\delta\overline{x_\infty}g) = \iota_*\delta \cdot \tilde{\iota}(\overline{x_\infty}) \cdot g = \gamma_0^r\delta^{-1}\gamma_\infty^s x_\infty g. \tag{4.12}$$

Combining (4.11) with (4.12), we get

$$\begin{aligned}
\overline{\alpha}([\overline{x_\infty}]) = [\delta\overline{x_\infty}g] \iff & \tilde{\iota}^{-1}([x_0])\\
&= [\tilde{\iota}^{-1}(\gamma_0^r\delta^{-1}\gamma_\infty^s x_\infty g)] \iff [x_0] = [\gamma_0^r\delta^{-1}\gamma_\infty^s x_\infty g] = [x_0 hg].
\end{aligned} \tag{4.13}$$

Here we have used the fact that $\tilde{\iota}$ commutes with $[\cdot]$. On the other hand, note that

$$G_{[\delta\overline{x_\infty}]} = G_{[\overline{x_\infty}]} = G_{[x_\infty]} = G_{[\delta x_0 h]} = G_{[x_0 h]} = h^{-1}G_{[x_0]}h.$$

Here we have used the fact that $\tilde{\iota}$ and (the fiber bijections induced by) δ are both G-equivariant. Thus (4.13) holds if and only if $hg \in G_{[x_0]}$, equivalently $g \in h^{-1}G_{[x_0]} = G_{[\delta\overline{x_\infty}]}h^{-1}$. This shows that

$$H_{\delta,\overline{x_\infty}} = G_{[\delta\overline{x_\infty}]}h^{-1}.$$

Thus $\mathrm{Inv}_\delta(\iota^*q, \iota^*\alpha) = [\![h^{-2}\varphi_{x_0}(\gamma_\infty^\delta)^{-1}h^2, h^{-1}]\!] = [\![\varphi_{x_0}(\gamma_\infty^\delta)^{-1}, h^{-1}]\!]$, which proves the proposition. ∎

4.10. *Automorphism groups of cuspidal objects of $\mathcal{A}dm(G)$.* According to Theorem 4.8.4, every object $(F, \alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}$ is isomorphic to an object of the form $(F_{u,h}, \alpha_{u,h})$, where (u, h) represents the class $\mathrm{Inv}_\delta(F, \alpha) \in \mathbb{I}(G)$. Here we calculate the automorphism group of the pair $(F_{u,h}, \alpha_{u,h})$.

First we consider automorphisms of the (Π, G)-set $F_{u,h}$. Let $M_{u,h} := \langle u, h^{-1}u^{-1}h\rangle$ as in (4.8). Let S_G denote the symmetric group on the underlying set of G. Then an automorphism of the (Π, G)-set $F_{u,h}$ is given by a permutation $\sigma \in S_G$ such that

- (G-equivariance) $\sigma(xg) = \sigma(x)g$ for all $x \in F_{u,h}$, $g \in G$;
- (Π-equivariance) $\sigma(mx) = m\sigma(x)$ for all $x \in F_{u,h}$, $m \in M_{u,h}$.

Equivariance in G implies that $\sigma(g) = \sigma(1 \cdot g) = \sigma(1) \cdot g$ so σ is determined by $\sigma(1) \in G$. On the other hand, Π-equivariance implies that $m\sigma(1) = \sigma(m \cdot 1) = \sigma(m) = \sigma(1 \cdot m) = \sigma(1) \cdot m$, so $\sigma(1) \in C_G(M_{u,h})$. Conversely, for any $a \in C_G(M_{u,h})$, the permutation

$$\sigma_a : G \to G \qquad \text{sending} \qquad g \mapsto ag$$

defines a permutation of $G = F_{u,h}$ which is both right G-equivariant and left $M_{u,h}$-equivariant, so it defines an automorphism of $F_{u,h}$. Next we seek to identify the elements $a \in C_G(M_{u,h})$ such that σ_a respects the gluing $\alpha_{u,h}$. For $a \in C_G(M_{u,h})$, σ_a respects the gluing if and only if we have an equality of cosets

$$\langle u \rangle \cdot h\sigma_a(x) = \langle u \rangle \cdot \sigma_a(hx) \qquad \text{for all } x \in F_{u,h}.$$

This is equivalent to saying

$$\langle u \rangle aha^{-1} = \langle u \rangle h \qquad \text{or equivalently} \qquad aha^{-1} = u^{k_a} h \quad \text{for some } k_a \in \mathbb{Z}. \tag{4.14}$$

Thus, automorphisms of $(F_{u,h}, \alpha_{u,h})$ are precisely the permutations σ_a such that $a \in G$ centralizes $M_{u,h} = \langle u, h^{-1}u^{-1}h \rangle$ and satisfies (4.14). Let

$$A_{G,u,h} := \{a \in C_G(M_{u,h}) : aha^{-1} = u^{k_a} h \text{ for some } k_a \in \mathbb{Z}\} \leq C_G(M_{u,h}). \tag{4.15}$$

PROPOSITION 4.10.1. *Let $A_{G,u,h}$ be as in (4.15). The map*

$$z : A_{G,u,h} \longrightarrow \langle u \rangle,$$
$$a \mapsto [a, h] := aha^{-1}h^{-1}$$

is a group homomorphism which fits into an exact sequence

$$1 \longrightarrow Z(G) \longrightarrow A_{G,u,h} \xrightarrow{z} \langle u^{k_{u,h}} \rangle \longrightarrow 1,$$

where $k_{u,h}$ is the smallest positive integer such that (u, h) is conjugate to $(u, u^{k_{u,h}} h)$. In particular, $A_{G,u,h}$ is an extension of a cyclic group of order $|u|/k_{u,h}$ by $Z(G)$. If $M_{u,h} = G$, then $A_{G,u,h} = Z(G)$ and $k_{u,h} = |u|$.

Remark 4.10.2. Let (u, h) correspond to the cuspidal G-cover $\pi : C \to E$ (Theorem 4.8.4). It follows from the equivalence $\mathcal{C}_E^{pc} \cong \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$ that $A_{G,u,h}$ is precisely the vertical automorphism group $\mathrm{Aut}^v(\pi)$ (viewing π in $\mathcal{A}dm(G)$). Certainly $k_{u,h}$ always divides $|u|$, and accordingly $Z(G)$ is always a subgroup of $A_{G,u,h} \cong \mathrm{Aut}^v(\pi)$. We have equality if and only if the generating pairs $(u, h), (u, uh), (u, u^2 h), \ldots, (u, u^{|u|-1}h)$ are all non-conjugate. Thus, equality fails if and only if there is some "unexpected" relation between these generating pairs, corresponding to an unexpected vertical automorphism that does not lie in $Z(G)$. In Section 5.3, we will show that for $G = \mathrm{SL}_2(\mathbb{F}_q)$ $(q \geq 5)$, there are no unexpected automorphisms, so $k_{u,h} = |u|$ and $A_{G,u,h} = Z(G)$ for any generating pair of $\mathrm{SL}_2(\mathbb{F}_q)$.

Proof of Proposition 4.10.1. The description of $A_{G,u,h}$ follows from the exactness of the sequence, so it suffices to establish exactness. If $a,b \in A_{G,u,h}$, then

$$z(ab) = abhb^{-1}a^{-1}h^{-1} = az(b)ha^{-1}h^{-1} = z(b)aha^{-1}h^{-1} = z(b)z(a) = z(a)z(b),$$

so z is a homomorphism. The kernel of z must commute with both h and $M_{u,h}$, but G is generated by $M_{u,h}$ and h, so the kernel is precisely $Z(G)$. Since $A_{G,u,h} \subset C_G(M_{u,h})$, if (u,h) is conjugate to $(u, u^{k_{u,h}}h)$, then there must be a $g \in G$ which centralizes u and satisfies ${}^g h := ghg^{-1} = u^{k_{u,h}}h$. But then

$${}^g(h^{-1}u^{-1}h) = {}^g h^{-1}\, {}^g u^{-1}\, {}^g h = h^{-1}u^{-k_{u,h}}u^{-1}u^{k_{u,h}}h = h^{-1}u^{-1}h,$$

so g also centralizes $h^{-1}u^{-1}h$, so we have $g \in C_G(M_{u,h})$, hence $g \in A_{G,u,h}$. This shows that z is surjective onto $\langle u^{k_{u,h}} \rangle$, so the sequence is exact. If $M_{u,h} = G$, then $C_G(M_{u,h}) = Z(G)$. Hence $A_{G,u,h}$ is both contained in and contains $Z(G)$, so it is equal to $Z(G)$. The exactness of the sequence then forces $k_{u,h} = |u|$. $\square$

Let ev_1 denote the "evaluation at 1" map

$$\mathrm{ev}_1 : \mathrm{Aut}_{\underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}}(F_{u,h}, \alpha_{u,h}) \longrightarrow G,$$
$$\sigma \mapsto \sigma(1).$$

THEOREM 4.10.3. *As usual we work over an algebraically closed field* k *of characteristic* 0. *Let* t_0 *be a tangential base point at* $0 \in \mathbb{P}^1$. *Let* ι *be the unique automorphism of* $\mathbb{P}^1$ *fixing* 1 *and swapping* $0, \infty$. *Let* $t_\infty := \iota(t_0)$, *and let* $\delta : t_0 \rightsquigarrow t_\infty$ *be a symmetric good path. The map* ev_1 *induces an isomorphism of* groups (*note that* $\sigma(\tau(1)) = \sigma(1 \cdot \tau(1)) = \sigma(1)\tau(1)$)

$$\mathrm{ev}_1 : \mathrm{Aut}_{\underline{\mathbf{Sets}}^{(\Pi,G)_{\delta,\succ}}}(F_{u,h}, \alpha_{u,h}) \xrightarrow{\sim} A_{G,u,h}.$$

Let $\pi : C \to E$ *be a cuspidal object of* $\mathcal{A}dm(G)(k)$ *with* $\mathrm{Inv}_\delta(\pi) = [\![u,h]\!]$. *Recall that its Higman invariant is* $[u^{-1}, h^{-1}]$ (*Proposition* 4.8.6). *Let* $\mathrm{Aut}^v(\pi)$ *denote its vertical automorphism group. This is precisely the group of* G*-equivariant automorphisms of* C *inducing the identity on* E. *For a generating pair* (u,h) *of* G, *let* $k_{u,h}$ *be the minimal positive integer such that* (u,h) *is conjugate to* $(u, u^{k_{u,h}}h)$. *Then*

(a) *Let* $A_{G,u,h}$ *be as in* (4.15). *There is an isomorphism*

$$\mathrm{Aut}^v(\pi) \xrightarrow{\sim} A_{G,u,h}.$$

Thus, the vertical automorphism group of π *is isomorphic to a subgroup of* G *which is an extension of a cyclic group of order* $|u|/k_{u,h}$ *by* $Z(G)$.

(a$'$) *The vertical automorphism groups of geometric points of* $\mathcal{A}dm(G)$ *are all reduced to* $Z(G)$ (*equivalently, the map* $\overline{\mathcal{M}(G)} \to \overline{\mathcal{M}(1)}$ *is representable*) *if and only if* $k_{u,h} = |u|$ *for all generating pairs* (u,h) *of* G.

(b) *If* $[\![u,h]\!] \neq [\![u^{-1},h^{-1}]\!]$, *then* $\mathrm{Aut}_{\mathcal{A}dm(G)(k)}(\pi) = \mathrm{Aut}^v(\pi)$. *If* $[\![u,h]\!] = [\![u^{-1},h^{-1}]\!]$, *then* $\mathrm{Aut}_{\mathcal{A}dm(G)(k)}(\pi)$ *is an extension of* $\mathrm{Aut}(E) \cong \mathbb{Z}/2\mathbb{Z}$ *by* $\mathrm{Aut}^v(\pi)$.

(c) *Every irreducible component of C is Galois over E with Galois group isomorphic to $M_{u,h} = \langle u, h^{-1}u^{-1}h\rangle$. The number of irreducible components of C is $[G : M_{u,h}]$. In particular, C is irreducible if and only if G is generated by $u, h^{-1}u^{-1}h$, in which case $\mathrm{Aut}^v(\pi) = Z(G)$.*

(d) *As usual, let $O \in E$ be the origin. Let $x \in \pi^{-1}(O)$. Then an automorphism $\sigma \in \mathrm{Aut}^v(\pi)$ fixes x if and only if it acts trivially on $\pi^{-1}(O)$, and there is an isomorphism*

$$\mathrm{Stab}_{\mathrm{Aut}^v(\pi)}(x) \xrightarrow{\sim} A_{G,u,h} \cap \langle [u^{-1}, h^{-1}]\rangle. \tag{4.16}$$

In particular, let $\pi : \mathcal{C} \to \mathcal{E}$ be the universal family over $\mathcal{A}dm(G)$ and let $\mathcal{R}_\pi$ be its reduced ramification divisor. Then for any cuspidal geometric point $z \in \mathcal{R}_\pi$ whose image in $\mathcal{A}dm(G)$ has δ-invariant $[\![u, h]\!]$, its vertical automorphism group is

$$\mathrm{Aut}^v(z) \cong A_{G,u,h} \cap \langle [u^{-1}, h^{-1}]\rangle. \tag{4.17}$$

Proof. Parts (a), (a′), and (c) follow immediately from Proposition 4.10.1 and the discussion above, noting that the map $\mathcal{A}dm(G) \to \overline{\mathcal{M}(G)}$ removes $Z(G)$ from all automorphism groups of geometric points.

For (b), note that since automorphisms of π in $\mathcal{A}dm(G)$ are G-equivariant, they descend to automorphisms of $E \cong C/G$, so we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^v(\pi) \longrightarrow \mathrm{Aut}_{\mathcal{A}dm(G)(k)}(\pi) \longrightarrow \mathrm{Aut}(E).$$

Since E is a nodal elliptic curve, $\mathrm{Aut}(E)$ is cyclic of order 2, so let $[-1]$ be the generator. Then the map $\mathrm{Aut}_{\mathcal{A}dm(G)(k)}(\pi) \to \mathrm{Aut}(E)$ is surjective if and only if there is an isomorphism $[-1]^*\pi \to \pi$ inducing the identity on E. By Proposition 4.9.1, this happens if and only if $[\![u, h]\!] = [\![u^{-1}, h^{-1}]\!]$.

It remains to prove (d). Since automorphisms of π are G-equivariant and G-acts transitively on all fibers, if an automorphism fixes a point, then it must fix the entire fiber. Next we prove the isomorphism (4.16). Let $\nu : \mathbb{P}^1 \to E$ be a standard normalization, and let $\Xi_\nu(\pi) = (\pi' : C' \to \mathbb{P}^1, \alpha_\pi)$ be the normalized-cover-with-gluing-data. Let $x' \in C'$ be the unique point lying over x, so $x' \in C'_1 := \pi'^{-1}(1)$. Let $\mathrm{Aut}_{\mathbb{P}^1}(\pi')$ be the automorphism group of $(\pi', \alpha_\pi) \in \mathcal{C}^{\succ}_{\mathbb{P}^1}$. Then by the equivalence $\Xi_\nu : \mathcal{C}^{pc}_E \xrightarrow{\sim} \mathcal{C}^{\succ}_{\mathbb{P}^1}$, it suffices to define an isomorphism

$$\mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}^1}(\pi')}(x') \xrightarrow{\sim} A_{G,u,h} \cap \langle u^{-1}h^{-1}uh\rangle.$$

Let t_1 be a tangential base point at $1 \in \mathbb{P}^1$, and let $\gamma_1 \in \pi_1(\mathbb{P}^*, t_1)$ be the canonical generator of inertia. Since δ is a good path, for some path $\epsilon : t_0 \rightsquigarrow t_1$, we have $(\gamma_\infty^\delta)^{-1}\gamma_0^{-1} = \gamma_1^\epsilon$. Let $\xi_1 : \langle\gamma_1\rangle\backslash C'_{t_1} \xrightarrow{\sim} C'_1$ be the canonical isomorphism given by the local picture near 1. Using the path ϵ, we obtain a G-equivariant bijection

$$\xi_1 \circ \epsilon : \langle\gamma_1^\epsilon\rangle\backslash C'_{t_0} \xrightarrow{\sim} \langle\gamma_1\rangle\backslash C'_{t_1} \xrightarrow{\sim} C'_1.$$

By Proposition 4.2.3(b), these bijections define natural isomorphisms of functors $\mathcal{C}_{\mathbb{P}^1}^{\succ} \to \underline{\mathbf{Sets}}$, and hence we can compute the action of $\mathrm{Aut}_{\mathbb{P}^1}(\pi')$ on C_1' via its action on $\langle\gamma_1^\epsilon\rangle\backslash C_{t_0}'$. Passing to $\underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$, let $F_\delta^{\succ}(\pi') = (C_{t_0}', \alpha) \in \underline{\mathbf{Sets}}^{(\Pi,G)_\delta,\succ}$. Then the same computations as above shows that for any choice of $x_0 \in C_{t_0}'$, $\mathrm{Aut}(C_{t_0}', \alpha) \cong A_{G,u,h}$, where $u = \varphi_{x_0}(\gamma_\infty^\delta)$, and $h \in H_{\delta,x_0}$. Since π is balanced, we also have $\varphi_{x_0}(\gamma_0) = h^{-1}u^{-1}h$. Here, for $a \in A_{G,u,h}$, let $\sigma_a \in \mathrm{Aut}(C_{t_0}', \alpha)$ be the corresponding automorphism. Then σ_a acts on C_{t_0}' by $\sigma_a(x_0 g) = x_0 a g$ for any $g \in G$. Since $\gamma_1^\epsilon = (\gamma_\infty^\delta)^{-1}\gamma_0^{-1}$, $\varphi_{x_0}(\gamma_1^\epsilon) = u^{-1}h^{-1}uh$, and for any $g \in G$, we have

$$\begin{aligned}
\sigma_a(\gamma_1^\epsilon x_0 g) &= \sigma_a(x_0\varphi_{x_0}(\gamma_1^\epsilon)g), \\
&= \sigma_a(x_0 u^{-1}h^{-1}uhg) \\
&= x_0(au^{-1}h^{-1}uhg),
\end{aligned}$$

so σ_a fixes the coset $\langle\gamma_1^\epsilon\rangle x_0 g$ if and only if $a \in \langle u^{-1}h^{-1}uh\rangle$. This establishes the isomorphism (4.16). The isomorphism (4.17) follows immediately from the definition of $\mathcal{R}_\pi$ in Section 3.2. $\square$

Remark 4.10.4. If G is abelian, then it follows from the discussion above that the vertical automorphism group of any cuspidal G-cover $\pi : C \to E$ is equal to G. In particular, the map $\overline{\mathcal{M}(G)} := \mathcal{A}dm(G) /\!\!/ Z(G) \to \overline{\mathcal{M}(1)}$ is representable. For $n \geq 3$, it is known that $\mathcal{M}(n) := \mathcal{M}(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$ is representable. On the other hand, it follows from Theorem 4.10.3(b) that for $n \geq 3$, the cuspidal objects of $\overline{\mathcal{M}(n)} := \overline{\mathcal{M}(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})}$ also have no automorphisms, so $\overline{\mathcal{M}(n)}$ is also representable (even as a stack over $\mathbb{Z}[1/n]$), and hence this gives another construction of the Deligne-Rapoport moduli stack of generalized elliptic curves with full level n structures [DR73, Th. 2.7].

Using this, we are able to give a purely combinatorial statement of Theorem 3.5.1.

THEOREM 4.10.5 (Combinatorial congruence). *Let G be a finite group, let $\mathfrak{c} \in \mathrm{Cl}(G)$ be a conjugacy class, and let $c \in \mathfrak{c}$ be a representative. Let $A_{G,u,h}$ be as in* (4.15).

(a) *Let $d' = d'_{\mathfrak{c}} := |C_G(\langle c\rangle)/\langle c\rangle|$.*
(b) *Let $m' = m'_{\mathfrak{c}}$ be the least positive integer which kills $A_{G,u,h} \cap \langle[u^{-1}, h^{-1}]\rangle$ for any generating pair (u, h) of G with $[u, h] \in \mathfrak{c}$.*

Then for any component $\mathcal{X} \subset \mathcal{A}dm(G)_{\mathfrak{c}}$ with coarse scheme X, the map to the j-line $X \to \overline{M(1)}$ satisfies

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \frac{|c|}{\gcd(|c|, m'd')}.$$

Combinatorially speaking, let $F_2 = \langle a, b\rangle$ *be a free group of rank* 2*, and let* $\gamma_{-I} \in \mathrm{Aut}^+(F_2)$ *be the automorphism* $(a,b) \mapsto (a^{-1}, b^{-1})$*. Note that the image of* γ_{-I} *in* $\mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$ *is central. Let* $\mathrm{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}} \subset \mathrm{Epi}^{\mathrm{ext}}(F_2, G)$ *be the subset represented by surjections* $\varphi : F_2 \twoheadrightarrow G$ *satisfying* $\varphi([a,b]) \in \mathfrak{c}$*. Then every* $\mathrm{Out}^+(F_2)$*-orbit on* $\mathrm{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}}/\langle\gamma_{-I}\rangle$ *has cardinality divisible by* $\frac{|c|}{\gcd(|c|, m'd')}$.[28]

Remark 4.10.6. Recall that if G is not 2-generated, then $\mathcal{A}dm(G)$ is empty (Corollary 2.5.4), so the theorem is only non-trivial for finite 2-generated groups G.

Proof. Let $\mathcal{C} \xrightarrow{\pi} \mathcal{E} \to \mathcal{A}dm(G)_{\mathfrak{c}}$ denote the universal family. Let $\mathcal{R}_\pi \subset \mathcal{C}$ denote the reduced ramification divisor. Suppose u, h is a generating pair of G with $[u,h] \in \mathfrak{c}$. By Proposition 3.5.2 the vertical automorphism groups of geometric points of $\mathcal{R}_\pi$ which lie over $\mathcal{M}(1)$ are equal to

$$Z(G) \cap \langle c\rangle = Z(G) \cap \langle [u^{-1}, h^{-1}]\rangle \subset A_{G,u,h} \cap \langle [u^{-1}, h^{-1}]\rangle$$

so they are all killed by m'. By Theorem 4.10.3(d), the vertical automorphism group of any geometric point of $\mathcal{R}_\pi$ is killed by m', so the full automorphism groups are killed by $12m'$. By Proposition 3.2.2, the components of $\mathcal{R}_\pi$ have degree over $\mathcal{X}$ dividing d', so Theorem 3.5.1 gives the congruence on degrees. For the congruence on $\mathrm{Out}^+(F_2)$-orbits, first by Proposition 2.5.10(b), $\mathcal{A}dm^0(G)$ is a gerbe over $\mathcal{M}(G)$, so X is the smooth compactification of a unique component $M \subset M(G)_{\mathfrak{c}}$. Thus, $\deg(X \to \overline{M(1)}) = \deg(M \to M(1))$. On the other hand, the fiber of the finite étale map $\mathcal{M}(G)_{\mathfrak{c}} \to \mathcal{M}(1)$ above a geometric point $x_E \in \mathcal{M}(1)$ is in bijection with $\mathrm{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}}$, and for a component $\mathcal{M} \subset \mathcal{M}(G)_{\mathfrak{c}}$, the fiber of $\mathcal{M} \to \mathcal{M}(1)$ is in bijection with an $\mathrm{Out}^+(F_2)$-orbit on $\mathrm{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}}$. By Theorem 2.5.2(4), any unramified geometric fiber of $M \to M(1)$ is in bijection with the quotient of a geometric fiber of $\mathcal{M} \to \mathcal{M}(1)$ by γ_{-I}. Thus we obtain the congruence on $\mathrm{Out}^+(F_2)$-orbits from the congruence on degrees. □

Remark 4.10.7. In Theorem 4.10.5, the full automorphism groups of geometric points of $\mathcal{R}_\pi$ can also be expressed totally combinatorially, but the payoff is limited to at most an improvement of the resulting congruence by a factor of 12, so we do not treat it here.

[28] Viewing F_2 as the fundamental group of a punctured torus and a, b as a positively oriented basis, then in the setup of Situation 2.5.14, the Higman invariant is technically given by $\varphi([b,a])$ (as opposed to $\varphi([a,b])$). However the automorphism of F_2 given by $(a,b) \mapsto (b,a)$ induces a bijection between the corresponding $\mathrm{Out}^+(F_2)$-orbits, and so the sizes of the orbits are the same.

4.11. *Necessity of the obstructions m, d.* Let $\mathcal{X} \subset \mathcal{A}dm(G)$ be a component classifying covers with ramification index e. Our main congruence (Theorem 3.5.1) gives

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \frac{12e}{\gcd(12e, m_{\mathcal{X}} d_{\mathcal{X}})},$$

which is possibly diluted by the integers $d_{\mathcal{X}}$ and $m_{\mathcal{X}}$. In the best possible case, we can obtain $\deg(X \to \overline{M(1)}) \equiv 0 \mod 12e$. However, in general, this is too much to hope for since, for example, the congruence modular curve $Adm(\mathbb{Z}/2\mathbb{Z}) = X_1(2)$ for elliptic curves with "$\Gamma_1(2)$-structures" corresponds to ramification index $e = 1$ but it has degree 3 over $\overline{M(1)}$; thus, the conditions on $m_{\mathcal{X}}, d_{\mathcal{X}}$ are necessary. A more reasonable question is to ask if we can always obtain the congruence

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod e. \tag{4.18}$$

It turns out this also does not hold in general, and both $m_{\mathcal{X}}, d_{\mathcal{X}}$ can be responsible for this failure. As an explicit example, let $D_{2k} := \mathbb{Z}/k\mathbb{Z} \rtimes \mu_2$ denote the dihedral group of order $2k$, where k is odd. In this case, its commutator subgroup is $\mathbb{Z}/k\mathbb{Z}$, its abelianization is μ_2, and its center is trivial. By [Che18, Th. 4.2.2], $Adm(D_{2k})$ is isomorphic to $\frac{\phi(k)}{2}$ copies of the curve $Adm(\mathbb{Z}/2\mathbb{Z})$. In particular, every component of $Adm(D_{2k})$ has degree 3 over $\overline{M(1)}$, whereas one can check that the commutator of any generating pair of D_{2k} has order k [Che18, §4.2], so every component of $Adm(D_{2k})$ classifies D_{2k}-covers with ramification index $e = k$. Letting $k \to \infty$, we see that in the general case, even the congruence (4.18) can fail arbitrarily badly. From the combinatorial congruence (Theorem 4.10.5), one can check that for $G = D_{2k}$, $d' = 2$, so $d_{\mathcal{X}} \mid 2$ and hence $d_{\mathcal{X}}$ does not affect the congruence. Thus the culprit must be $m_{\mathcal{X}}$. Indeed, let (u, h) be the generating pair

$$(u, h) = ((1, 1), (0, -1)) \in D_{2k} = \mathbb{Z}/k\mathbb{Z} \rtimes \mu_2.$$

In this case, we find that (u, h) is conjugate to (u, uh), so in this case we have that $A_{D_{2k},u,h} = [D_{2k}, D_{2k}]$ has order k, so the vertical automorphism group of the cusp in $\mathcal{A}dm(D_{2k})$ with δ-invariant $[\![u, h]\!]$ has order k, which implies that $k \mid m_{\mathcal{X}}$. Thus in this case $m_{\mathcal{X}}$ is totally responsible for the failure of the congruence.

Let $\mathrm{U}_3(\mathbb{F}_4)$ denote the group of isometries of a 3-dimensional Hermitian space over $\mathbb{F}_{16}$ relative to the involution of $\mathbb{F}_{16}/\mathbb{F}_4$, let $\mathrm{SU}_3(\mathbb{F}_4)$ be the subgroup of isometries of determinant 1, and let $\mathrm{PSU}_3(\mathbb{F}_4)$ denote its quotient by the center. Then $\mathrm{PSU}_3(\mathbb{F}_4)$ is a finite simple group of order 62400. It has six conjugacy classes of elements of order 5, four of which are powers of each other.[29] We have computed using the computer algebra package GAP that for

[29] In other words, these four conjugacy classes all generate the same rational class.

any one of these four classes $\mathfrak{c}$, $m'_{\mathfrak{c}} = 1$ (see Theorem 4.10.5). It follows that $m_{\mathcal{X}} \mid 12$ for any component $\mathcal{X} \subset \mathcal{A}dm(\mathrm{PSU}_3(\mathbb{F}_4))_{\mathfrak{c}}$. On the other hand, from our computations $\mathcal{A}dm(\mathrm{PSU}_3(\mathbb{F}_4))_{\mathfrak{c}}$ has two components, of degrees 1 and 184 respectively over $\mathcal{M}(1)$. It follows that in this case $d_{\mathcal{X}}$ must be responsible for the failure of the congruence $\equiv 0 \mod 5$.

4.12. *Congruences for non-abelian finite simple groups.* If $f : G_1 \to G_2$ is a surjection of 2-generated finite groups, then by Theorem 2.5.2(6), f induces a finite etale surjection of moduli spaces $\mathcal{M}(G_1) \to \mathcal{M}(G_2)$ (corresponding to the surjection of sets $f_* : \mathrm{Epi}^{\mathrm{ext}}(F_2, G_1) \to \mathrm{Epi}^{\mathrm{ext}}(F_2, G_2)$). Thus, in terms of understanding the sizes of $\mathrm{Out}^+(F_2)$-orbits, a crucial special case is when G is a finite simple group. If G is cyclic, then $\mathrm{Out}^+(F_2)$ acts transitively on $\mathrm{Epi}^{\mathrm{ext}}(F_2, G)$ with size equal to the index of the congruence subgroup $\Gamma_1(n) \le \mathrm{SL}_2(\mathbb{Z})$ (cf. [DS05, §3.9]). Moreover, note that in this case our results say nothing — when G is abelian, $|c| = 1$ and hence the congruence in Theorem 4.10.5 is trivial.

In this section, we will show that Theorem 4.10.5 often yields non-trivial congruences when G is a non-abelian finite simple group. For this we wish to control the integers $d'_{\mathfrak{c}}$ and $m'_{\mathfrak{c}}$ of Theorem 4.10.5. More generally, we also obtain non-trivial congruences if we have some control on the proper normal subgroups of G. We begin with some group theoretic lemmas:

Lemma 4.12.1. *Let G be a group and $U \le G$ be a cyclic subgroup of finite index. Then for any $g \in G$, g normalizes $U \cap gUg^{-1}$.*

Proof. Conjugation by g maps $U \cap gUg^{-1}$ to $gUg^{-1} \cap g^2Ug^{-2}$, but these are both subgroups of the cyclic group gUg^{-1} of the same order, so they are equal. Thus g normalizes $U \cap gUg^{-1}$. ☐

Lemma 4.12.2. *Let G be a non-abelian finite group. In the notation of Theorem 4.10.5, let ℓ be a prime and let $k, j \ge 0$ be integers such that*

(a) $k := \mathrm{ord}_\ell(m'_{\mathfrak{c}})$, *and*[30]
(b) *G does not contain a proper normal subgroup of order divisible by ℓ^{j+1}.*

If $k \ge j$, then G must contain a subgroup isomorphic to $\mathbb{Z}/\ell^k\mathbb{Z} \times \mathbb{Z}/\ell^{k-j}\mathbb{Z}$. In particular, we must have $\ell^{2k-j} \mid |G|$.

Proof. By Theorem 4.10.5, there must exist a generating pair (u, h) of G with $[u, h] \in \mathfrak{c}$ such that there exists an element $z \in A_{G,u,h} \cap \langle [u^{-1}, h^{-1}] \rangle$ of order ℓ^k. By Proposition 4.10.1, $|A_{G,u,h}|$ divides $|u| \cdot |Z(G)|$. Since G is

[30]Note that if G cannot be generated by two elements, then $m'_{\mathfrak{c}}$ is the least positive integer which satisfies a trivial condition, hence $m'_{\mathfrak{c}} = 1$. In particular this lemma implicitly assumes that G can be generated by two elements.

non-abelian, (b) implies that ℓ^{k-j} divides $|u|$. Moreover z centralizes $M_{u,h} = \langle u, h^{-1}u^{-1}h\rangle$. Let $U := \langle u\rangle$ and $U^h := \langle h^{-1}uh\rangle$. By Lemma 4.12.1, h normalizes $U \cap U^h$. Since U is cyclic, u normalizes $U \cap U^h$, so $G = \langle u, h\rangle$ also normalizes $U \cap U^h$. Since G is not cyclic, (b) implies that ℓ^{j+1} does not divide $|U \cap U^h|$. Thus $|\langle z\rangle \cap U|$ and $|\langle z\rangle \cap U^h|$ cannot both be divisible by ℓ^{j+1}. Note that since z centralizes $M_{u,h}$, both $\langle z, u\rangle$ and $\langle z, u^h\rangle$ are abelian subgroups of G. For simplicity, assume ℓ^{j+1} does not divide $|\langle z\rangle \cap U|$. Choose $z' \in \langle z\rangle, u' \in \langle u\rangle$ such that $|z'| = \ell^k$ and $|u'| = \ell^{k-j}$. Then ℓ^{j+1} does not divide $|\langle z'\rangle \cap \langle u'\rangle|$, so $\langle z'^j\rangle \cap \langle u'\rangle$ is trivial, so $\langle z'^j, u'\rangle \cong (\mathbb{Z}/\ell^{k-j}\mathbb{Z})^2$. It follows that $\langle z', u'\rangle \cong \mathbb{Z}/\ell^k\mathbb{Z} \times \mathbb{Z}/\ell^{k-j}\mathbb{Z}$. □

The ℓ-adic valuation $\mathrm{ord}_\ell(d'_\mathfrak{c})$ can be easily bounded if we know the relative valuations of $\mathrm{ord}_\ell(|c|)$ and $\mathrm{ord}_\ell(|G|)$. One use of Lemma 4.12.2 is to do the same for $\mathrm{ord}_\ell(m'_\mathfrak{c})$. A theorem of Vdovin gives another method to control $d'_\mathfrak{c}, m'_\mathfrak{c}$ when G is a non-abelian finite simple group other than $\mathrm{PSL}_2(\mathbb{F}_q)$:

Theorem 4.12.3 (Vdovin [Vdo99, Th. A]). *If G is a non-abelian finite simple group which is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ (for any prime power q), then for any abelian subgroup $A \le G$, we have $|A| < |G|^{1/3}$.*

To summarize, $d'_\mathfrak{c}$ and $m'_\mathfrak{c}$ can be controlled as follows.

Corollary 4.12.4. *Let G be a finite group. Let $c \in G$, and let $\mathfrak{c}$ be its conjugacy class. Let $\mathcal{X} \subset \mathcal{A}dm(G)_\mathfrak{c}$ be a connected component with coarse scheme X. For a prime ℓ, let $r := \mathrm{ord}_\ell(|c|)$. Let $d'_\mathfrak{c}, m'_\mathfrak{c}$ be as in Theorem 4.10.5.*

(a) *Suppose G is non-abelian. Write $\mathrm{ord}_\ell(|G|) = r + s$, and let $j \ge 0$ be an integer such that G does not contain any proper normal subgroups of order divisible by ℓ^{j+1}. Then*
   - $\mathrm{ord}_\ell(d'_\mathfrak{c}) \le s$;
   - $\mathrm{ord}_\ell(m'_\mathfrak{c}) \le \lfloor \frac{r+s+j}{2} \rfloor$.

(b) *Suppose G is non-abelian and simple.*
   - *If $\ell^{r+1} \ge |G|^{1/3}$ and G is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ for any q, then* $\mathrm{ord}_\ell(d'_\mathfrak{c}) = 0$.
   - *If $\ell^{k+1} \ge |G|^{1/3}$ ($k \in \mathbb{Z}$) and G is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ for any q, then $\mathrm{ord}_\ell(m'_\mathfrak{c}) \le \lfloor \frac{k}{2} \rfloor$.*

Proof. Part (a) follows from Lemma 4.12.2. For (b), if $\mathrm{ord}_\ell(d'_\mathfrak{c}) > 0$, then there exists an ℓ-power torsion element $z \in C_G(\langle c\rangle)$ which does not lie in $\langle c\rangle$. Then $\langle z, c\rangle$ is abelian of order ℓ^{r+1}, which is forbidden by Vdovin's theorem. Similarly, (c) immediately follows from Lemma 4.12.2 and Vdovin's theorem. □

This allows us to guarantee a non-trivial congruence in a number of settings. For example, we have

Corollary 4.12.5. *Let G be a finite group. Let $c \in G$, and let $\mathfrak{c}$ be its conjugacy class. Let $\mathcal{X} \subset \mathcal{A}dm(G)_{\mathfrak{c}}$ be a connected component with coarse scheme X. For a prime ℓ, let $r := \operatorname{ord}_\ell(|c|)$. Then we have*

(a) *Write $\operatorname{ord}_\ell(|G|) = r + s$, and let $j \geq 0$ be an integer such that G does not contain any proper normal subgroup of order divisible by ℓ^{j+1}. Then*

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \ell^{\lceil \frac{r-3s-j}{2} \rceil}.$$

Combinatorially speaking, using the notation of 4.10.5, every $\operatorname{Out}^+(F_2)$-orbit on $\operatorname{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}}/\langle \gamma_{-I} \rangle$ has cardinality divisible by $\ell^{\lceil \frac{r-3s-j}{2} \rceil}$.

(b) *Suppose G is non-abelian and simple. If $\ell^{r+1} \geq |G|^{1/3}$ and G is not isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ for any q, then*

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \ell^{\lceil \frac{r}{2} \rceil}.$$

Combinatorially speaking, using the notation of 4.10.5, every $\operatorname{Out}^+(F_2)$-orbit on $\operatorname{Epi}^{\mathrm{ext}}(F_2, G)_{\mathfrak{c}}/\langle \gamma_{-I} \rangle$ has cardinality divisible by $\ell^{\lceil \frac{r}{2} \rceil}$.

Proof. We note that if G is abelian, then $\mathcal{A}dm(G)_{\mathfrak{c}}$ is empty for any nontrivial conjugacy class $\mathfrak{c}$, so (a) holds trivially in this case (see Remark 2.3.3). If G is non-abelian, (a) and (b) follow immediately from Corollary 4.12.4 and Theorem 4.10.5. $\square$

Remark 4.12.6. We make a few observations.

- If $\ell \geq 3$ and $\mathfrak{c}$ is a class of $\mathrm{SL}_2(\mathbb{F}_\ell)$ of order divisible by ℓ, then since $\ell^2 \nmid |\mathrm{SL}_2(\mathbb{F}_\ell)|$, we may apply Corollary 4.12.5(a) with $r = 1$, $s = j = 0$ to obtain a congruence mod ℓ. We will later recover this fact from a more general analysis in Section 5.3 below, where we will even show that when $G = \mathrm{SL}_2(\mathbb{F}_q)$ and $q \geq 3$, $A_{\mathrm{SL}_2(\mathbb{F}_q),u,h} = Z(\mathrm{SL}_2(\mathbb{F}_q))$ for any generating pair of $\mathrm{SL}_2(\mathbb{F}_q)$. Moreover we will explicitly compute $d'_{\mathfrak{c}}$ and $m'_{\mathfrak{c}}$ for any Higman invariant $\mathfrak{c}$.
- Lemma 4.12.2 can be slightly sharpened by weakening condition (b). The proof only requires that ℓ^{j+1} does not divide $|Z(G)|$ or the order of any proper normal *cyclic* subgroup. Accordingly part (a) of the above corollaries can also be sharpened.
- Part (b) of the above corollaries use Vdovin's theorem and hence can be slightly sharpened by noting that the proof of Lemma 4.12.2 often gives an abelian subgroup which is larger than $(\mathbb{Z}/\ell^k\mathbb{Z})^2$.

5. Applications to Markoff triples and the geometry of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))$

In this section we will specialize our discussions above to the case of admissible $\mathrm{SL}_2(\mathbb{F}_q)$-covers of elliptic curves. The key features that become available in this special case are the explicit description of the character variety for SL_2-representations of a free group of rank 2, which is explained in Section 5.2, and the work of Bourgain, Gamburd, and Sarnak as explained in Section 5.5. In Section 5.3, we use the theory of this character variety and the results obtained in Section 4.10 to compute the vertical automorphism groups of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))$; we will prove that these automorphism groups are reduced to the center of $\mathrm{SL}_2(\mathbb{F}_q)$. In Section 5.4, we put everything together to obtain congruences for many Markoff-type equations. The application to the Markoff equation $x^2+y^2+z^2-3xyz=0$ is explicitly described in Section 5.5. In Section 5.6 we give a genus formula for the components of $M(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$.

We work universally over a base scheme $\mathbb{S}$ over which $|G|$ is invertible.

5.1. *The trace invariant.* If $\pi : C \to E$ is an admissible G-cover of a 1-generalized elliptic curve E over an algebraically closed field k, then relative to a compatible system of roots of unity $\{\zeta_n\}_{n\geq 1}$, we have defined its Higman invariant in Section 2.3, which is a conjugacy class in G.

Definition 5.1.1. Let $q = p^r$ be a prime power. Let $G \leq \mathrm{GL}_2(\mathbb{F}_q)$ be a subgroup. Then the trace invariant of a geometric point of $\mathcal{A}dm(G)_{\overline{\mathbb{Q}}}$ is the trace of its Higman invariant relative to $\{\exp(\frac{2\pi i}{n})\}_{n\geq 1}$.

As in Section 2.3, we also have decompositions

$$\begin{aligned}\overline{\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))}_{\overline{\mathbb{Q}}} &= \bigsqcup_{t\in\mathbb{F}_q} \overline{\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))}_t \quad \text{and} \\ \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\overline{\mathbb{Q}}} &= \bigsqcup_{t\in\mathbb{F}_q} \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_t\end{aligned} \tag{5.1}$$

into open and closed substacks corresponding to objects with trace invariant t.

Definition 5.1.2. We say that a conjugacy class $\mathfrak{c} \in \mathrm{Cl}(\mathrm{SL}_2(\mathbb{F}_q))$ (resp. an element $t \in \mathbb{F}_q$) is q-admissible if it is the conjugacy class (resp. trace) of the commutator of a generating pair of $\mathrm{SL}_2(\mathbb{F}_q)$.

The q-admissible classes (resp. traces) were classified by McCullough and Wanderley [MW11].

PROPOSITION 5.1.3. *We state the following results in pairs, beginning with a group-theoretic statement, followed by a geometric consequence.*

(a) *If $q = 2, 4, 8$ or $q \geq 13$, then the q-admissible traces are $\mathbb{F}_q - \{2\}$. If $q = 3, 9, 11$, then the q-admissible traces are $\mathbb{F}_q - \{1, 2\}$. If $q = 5$, the q-admissible traces are $\in \mathbb{F}_q - \{0, 2, 4\}$. If $q = 7$, the q-admissible traces are $\in \mathbb{F}_q - \{0, 1, 2\}$.*

(a′) *For any $t \in \mathbb{F}_q$, $\overline{\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))}_t$ is non-empty if and only if t is q-admissible.*

(b) *If q is even, then for any $t \in \mathbb{F}_q - \{\pm 2\} = \mathbb{F}_q - \{0\}$, there exists a unique conjugacy class in $\mathrm{SL}_2(\mathbb{F}_q)$ with trace t. If $t = \pm 2 = 0$, then there exist precisely two classes with trace t, represented by $\left[\begin{smallmatrix}\pm 1 & 0\\ 0 & \pm 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\pm 1 & 1\\ 0 & \pm 1\end{smallmatrix}\right]$, neither of which are q-admissible. In particular, every q-admissible trace*

$t \in \mathbb{F}_q$ is the trace of a unique *q-admissible class $\mathfrak{c}$, and for any $a \in \mathbb{F}_q$, any non-central element of $\mathrm{SL}_2(\mathbb{F}_q)$ with trace a has the same order.*

(b′) *If q is even, then for any q-admissible class $\mathfrak{c}$ with trace t, the map*

$$\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}} \longrightarrow \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_t$$

is an isomorphism.

(c) *If q is odd, then for any $t \in \mathbb{F}_q - \{\pm 2\}$, there exists a unique conjugacy class in $\mathrm{SL}_2(\mathbb{F}_q)$ with trace t. If $t = \pm 2$, then there exist precisely three classes with trace t, represented by $\left[\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} \pm 1 & a \\ 0 & \pm 1 \end{smallmatrix}\right]$, where $a \in \mathbb{F}_q^\times$ is not a square. In particular, every q-admissible t other than -2 is the trace of a* unique *q-admissible class, and for any $a \in \mathbb{F}_q$, any non-central element of $\mathrm{SL}_2(\mathbb{F}_q)$ with trace a has the same order. The q-admissible classes of trace -2 are represented by $\left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & a \\ 0 & -1 \end{smallmatrix}\right]$ where $a \in \mathbb{F}_q^\times$ is a non-square. If $u \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$ with $u^2 \in \mathbb{F}_q$ and $\gamma := \left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right]$, then conjugation by γ switches the two q-admissible classes of trace -2.*

(c′) *If q is odd, then for any q-admissible class $\mathfrak{c}$ with trace t, the map*

$$\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}} \longrightarrow \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_t$$

is an isomorphism except for $t = -2$. If $t = -2$, let $\mathfrak{c}_1, \mathfrak{c}_2$ denote the two q-admissible classes of trace -2. Then conjugation by γ induces an isomorphism $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}_1} \xrightarrow{\sim} \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}_2}$, and

$$\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{-2} = \bigsqcup_{\mathfrak{c} \in \{\mathfrak{c}_1, \mathfrak{c}_2\}} \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}}.$$

Proof. Part (a) is precisely [MW11, Th. 2.1]. Parts (b) and (c) follow from [MW13, §5]. Parts (a′) and (b′) follow immediately from (a) and (b). For part (c′), assume q odd, and let $\mathfrak{c}_1, \mathfrak{c}_2$ denote the two q-admissible classes of trace invariant -2. Let $i_\gamma \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_q))$ be given by $g \mapsto \gamma g \gamma^{-1}$, then i_γ defines a map

$$\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q)) \longrightarrow \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))$$

sending an admissible G-cover $\pi : C \to E$ to the same map $\pi : C \to E$, but with G-action defined via the isomorphism i_γ. Since i_γ has finite order, this is an equivalence. Since $u^2 \in \mathbb{F}_q^\times$ is a non-square, i_γ switches the two q-admissible classes of trace invariant -2, and hence it restricts to an equivalence

$$\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}_1} \xrightarrow{\sim} \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\mathfrak{c}_2}. \qquad \square$$

Let $I := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. By Proposition 5.1.3(b) and (c), for any $a \in \mathbb{F}_q$, with the exception of the conjugacy classes of $\pm I$, the conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_q)$ with trace a all have the same order. Thus, we may define

Definition 5.1.4. Let $q = p^r$ be a prime power. For $a \in \mathbb{F}_q$, let $n_q(a)$ denote the order of any matrix $A \in \mathrm{SL}_2(\overline{\mathbb{F}_q}) - \{\pm I\}$ with trace a.

PROPOSITION 5.1.5. *Let $q = p^r$ be a prime power. For $a \in \mathbb{F}_q$, there is a set $\{\omega, \omega^{-1}\} \subset \mathbb{F}_{q^2}$ which is uniquely determined by the property that $a = \omega + \omega^{-1}$. The integer $n_q(a)$ satisfies*

$$n_q(a) = \begin{cases} 2 & \text{if } q \text{ is even and } a = \pm 2 = 0, \\ p & \text{if } q \text{ is odd and } a = 2, \\ 2p & \text{if } q \text{ is odd and } a = -2, \\ |\omega| & \text{if } a \neq \pm 2. \end{cases}$$

Proof. If $\omega + \omega^{-1} = a$, then $\{\omega, \omega^{-1}\}$ are precisely the roots of the polynomial $x^2 - ax + 1$, so they are determined by a. The description of $n_q(a)$ follows from Proposition 5.1.3, noting that if $a \neq \pm 2$, then $\omega \neq \omega^{-1}$ so any matrix with trace a is diagonalizable over $\mathbb{F}_{q^2}$. □

5.2. $\mathrm{SL}_2(\mathbb{F}_q)$*-structures as* $\mathbb{F}_q$*-points of a character variety.* Let G be a finite group. Let $E, \Pi, a, b, x_E, \alpha$ be as in Situation 2.5.14. Then the fiber of $\mathcal{M}(G)_{\overline{\mathbb{Q}}}/\mathcal{M}(1)_{\overline{\mathbb{Q}}}$ over x_E is canonically in bijection with

$$\mathrm{Epi}^{\mathrm{ext}}(\Pi, G).$$

Under this bijection, the orbits of the $\mathrm{Out}^+(\Pi)$-action correspond to the components of $\mathcal{M}(G)_{\overline{\mathbb{Q}}}$. In this section we will show that when $G = \mathrm{SL}_2(\mathbb{F}_q)$, $\mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))$ is closely related to the $\mathbb{F}_q$ points of a certain character variety. This correspondence is central in our geometric approach to the conjecture of Bourgain, Gamburd, and Sarnak 1.2.2, and moreover it allows us to calculate the automorphism groups of geometric points of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))$, which forms the key input to Theorem 3.5.1.

Using the oriented basis $a, b \in \Pi$, there is a bijection

$$\begin{array}{rcl} \mathrm{Epi}(\Pi, G) & \xrightarrow{\sim} & \{(A, B) \mid A, B \text{ generate } G\}, \\ \varphi & \mapsto & (\varphi(a), \varphi(b)). \end{array} \tag{5.2}$$

For the purposes of understanding the $\mathrm{Out}^+(\Pi)$-orbits, it suffices to consider the orbits of $\mathrm{Aut}^+(\Pi)$ on the set of generating pairs of G. An elementary Nielsen move applied to a generating pair (A, B) of G sends (A, B) to any of

$$(A, AB), (B, A), \text{ or } (A, B^{-1}). \tag{5.3}$$

Two generating pairs of G are said to be Nielsen equivalent if they are related by a sequence of elementary Nielsen moves. One can check that the elementary Nielsen moves, applied to the generators (a, b) of Π, define a set of generators of $\mathrm{Aut}(\Pi)$ [MKS04, §3], and that two generating pairs of G are Nielsen equivalent if and only if they lie in the same $\mathrm{Aut}(\Pi)$-orbit via the bijection in (5.2). Questions about Nielsen equivalence were traditionally studied from the point of view of combinatorial group theory [Pak01, §2]. However, if R is a ring, S an R-algebra, and G is obtained as the S-points of an algebraic group $\mathcal{G}/R$, then the set $\mathrm{Hom}(\Pi, G)$ can be viewed as the S-points of the functor $\mathrm{Hom}(\Pi, \mathcal{G}) : \underline{\mathbf{Sch}}/R \to \underline{\mathbf{Sets}}$ sending $T \mapsto \mathrm{Hom}(\Pi, \mathcal{G}(T))$. Since Π is free on a, b, this functor is representable by the scheme $\mathcal{G} \times \mathcal{G}$, and hence $\mathrm{Epi}(\Pi, G)$ inherits an algebraic structure as a subset of the S-points of $\mathcal{G} \times \mathcal{G}$. Similarly $\mathrm{Epi}^{\mathrm{ext}}(\Pi, G)$ in many cases can be approximated by a subset of the S-points of

$$\mathrm{Hom}(\Pi, \mathcal{G})/\mathcal{G} = (\mathcal{G} \times \mathcal{G})/\mathcal{G},$$

where the action of $\mathcal{G}$ is by simultaneous conjugation, and the quotient is taken in a suitable sense (see Theorem 5.2.10 and Remark 5.2.11 below). Moreover, since $\mathrm{Aut}(\Pi)$ is generated by automorphisms of the form (5.3), $\mathrm{Aut}(\Pi)$ also acts on $\mathrm{Hom}(\Pi, \mathcal{G})$ and $\mathrm{Hom}(\Pi, \mathcal{G})/\mathcal{G}$ (on the right!) as automorphisms of schemes. When $\mathcal{G} = \mathrm{SL}_2$, these observations together with the work of Brumfiel-Hilden [BH95] and Nakamoto [Nak00] provide the connection between the sets $\mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))$ and $\mathbb{F}_q$-points of the Markoff equation (Theorem 5.2.10). In Section 5.3, we will also use this relationship to compute the vertical automorphism groups of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q))$.

When $\mathcal{G} = \mathrm{SL}_{2,R}$, the center of $\mathrm{GL}_{2,R}$ acts trivially on $\mathrm{Hom}(\Pi, \mathcal{G})$, so taking quotients by $\mathrm{SL}_{2,R}, \mathrm{GL}_{2,R}$ or $\mathrm{PGL}_{2,R}$ are all the same. For the purposes of our exposition it will be convenient to consider the quotient by $\mathrm{GL}_{2,R}$.

THEOREM 5.2.1. *Let R be any ring. Let $A[\Pi] = A[\Pi]_R$ denote the affine ring of* $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R}) \cong \mathrm{SL}_{2,R} \times \mathrm{SL}_{2,R}$. *Let $X_{\mathrm{SL}_{2,R}}$ be the quotient*

$$\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R}) /\!\!/ \mathrm{GL}_{2,R} := \mathrm{Spec}\, A[\Pi]^{\mathrm{GL}_{2,R}},$$

where $\mathrm{GL}_{2,R}$ *acts via conjugation on* $\mathrm{SL}_{2,R}$. *If $R = \mathbb{Z}$, we will simply write $X_{\mathrm{SL}_2} := X_{\mathrm{SL}_{2,\mathbb{Z}}}$. Let* Tr *denote the map*

$$\mathrm{Tr} : \mathrm{Hom}(\Pi, \mathrm{SL}_{2,R}) \longrightarrow \mathbb{A}^3_R$$

defined on A-valued points[31] *for various R-algebras A by sending $\varphi : \Pi \to$* $\mathrm{SL}_2(A)$ *to* $\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b)$, *and* $\mathrm{tr}\,\varphi(ab)$ *respectively. Then* Tr *induces an isomorphism* (*which we also denote by* Tr)

$$\mathrm{Tr} : X_{\mathrm{SL}_{2,R}} \xrightarrow{\sim} \mathbb{A}^3_R. \tag{5.4}$$

In particular, formation of the quotient commutes with arbitrary base change in R.

Proof. Let A, B, C denote the functions $\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab) \in A[\Pi]$ respectively. We must show that for any ring R, $A[\Pi]^{\mathrm{GL}_{2,R}} = R[A, B, C]$. When $R = \mathbb{C}$ this amounts to a classical result of Fricke and Vogt (see [Gol03], [Gol09]). The general case is treated in Brumfiel-Hilden [BH95, Props. 3.5 and

[31] Since every scheme is covered by affine opens, to define a morphism of schemes, it suffices to define it on T-valued points for all affine schemes T.

9.1(ii)]; however they do not distinguish between $\mathrm{GL}_2(R)$ and $\mathrm{GL}_{2,R}$, so their exposition is not complete when $\mathrm{GL}_2(R)$ is not Zariski dense in $\mathrm{GL}_{2,R}$. If we assume their result when we have Zariski density (e.g., when $R = \mathbb{Z}$ or R is an infinite field), then the general case can be deduced by reducing to known cases using the universal coefficient theorem [Jan03, I, Prop. 4.18]. For completeness we also give a proof in the appendix, Section 6.2, following [BH95]. $\square$

In fact the map $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R}) \to X_{\mathrm{SL}_{2,R}}$ is a uniform categorical quotient in the sense of geometric invariant theory, though we will not need this [Ses77, Rem. 8] (also see [MFK94, Th. 1.1]).

Definition 5.2.2. With notation as in Theorem 5.2.1, $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})$ is the *representation variety* for $\mathrm{SL}_{2,R}$-representations of Π, and $X_{\mathrm{SL}_{2,R}}$ is the *character variety* for $\mathrm{SL}_{2,R}$-representations of Π. Since formation of $X_{\mathrm{SL}_{2,R}}$ commutes with base change in R, for most applications it will suffice to work with $X_{\mathrm{SL}_2} = X_{\mathrm{SL}_{2,\mathbb{Z}}}$.

Definition 5.2.3. For a ring A and $\varphi \in \mathrm{Hom}(\Pi, \mathrm{SL}_2(A))$, we will call $\mathrm{Tr}(\varphi) \in \mathbb{A}^3(A) = A^3$ the "trace coordinates of φ." The *trace invariant* of φ is the element $\mathrm{tr}\,\varphi([b,a]) = \mathrm{tr}\,\varphi([a,b]) \in A$.

There is a natural right action of $\mathrm{Aut}(\Pi)$ on the functor $\mathrm{Hom}(\Pi, \mathrm{SL}_2)$ commuting with the conjugation action of GL_2. By the Yoneda lemma, it follows that $\mathrm{Aut}(\Pi)$ acts on trace coordinates by *polynomials*, and the action descends to a right action of $\mathrm{Aut}(\Pi)$ on the character variety $X_{\mathrm{SL}_2} \cong \mathbb{A}^3$. Moreover, we will see that this action preserves the trace invariant (Proposition 5.2.5 below).

Lemma 5.2.4. *Let R be any ring, and let Π be a free group with generators a, b. Let $\varphi : \Pi \to \mathrm{SL}_2(R)$ be a homomorphism. Let $(A, B, C) := (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab))$. Then the trace invariant of φ can be computed as follows*:

$$\mathrm{tr}(\varphi([a,b])) = A^2 + B^2 + C^2 - ABC - 2 \in R.$$

Proof. See [BH95, Prop. A.1*.10(iii)], and also [Gol09, §2.2]. $\square$

Proposition 5.2.5. *Let $T : \mathbb{A}^3 \to \mathbb{A}^1$ (over $\mathbb{Z}$) be given by*

$$T(x,y,z) = x^2 + y^2 + z^2 - xyz - 2.$$

Let $\tau : X_{\mathrm{SL}_2} \to \mathbb{A}^1$ be given by $\varphi \mapsto \mathrm{tr}\,\varphi([a,b])$. Then the following diagram is commutative:

$$\begin{tikzcd} X_{\mathrm{SL}_2} \arrow[r, "\mathrm{Tr}"] \arrow[dr, "\tau"'] & \mathbb{A}^3 \arrow[d, "T"] \\ & \mathbb{A}^1, \end{tikzcd}$$

where Tr *is the isomorphism* (5.4). *The right action of* $\mathrm{Aut}(\Pi)$ *on* X_{SL_2} *and its induced action on* $\mathbb{A}^3$ *preserve the fibers of* τ *and* T. *Viewing the induced action on* $\mathbb{A}^3$ *as a left action,* Tr *defines an* anti-homomorphism

$$\mathrm{Tr}_* : \mathrm{Aut}(\Pi) \hookrightarrow \mathrm{Aut}(\mathbb{A}^3),$$

which can be described as follows. The group $\mathrm{Aut}(\Pi)$ *is generated by three elements* r, s, t *(defined below), and their images in* $\mathrm{Aut}(\mathbb{A}^3)$ *under the anti-homomorphism* Tr_* *are given as follows*:

$$\begin{array}{rclcrcl} r:(a,b) & \mapsto & (a^{-1},b) & & R_3:(x,y,z) & \mapsto & (x,y,xy-z), \\ s:(a,b) & \mapsto & (b,a) & \xrightarrow{\mathrm{Tr}_*} & \tau_{12}:(x,y,z) & \mapsto & (y,x,z), \\ t:(a,b) & \mapsto & (a^{-1},ab) & & \tau_{23}:(x,y,z) & \mapsto & (x,z,y). \end{array}$$

Proof. We begin with the final statement. That r, s, t generate $\mathrm{Aut}(\Pi)$ follows from [MKS04, §3] (also see [MW13, §2]). That $(s,t) \mapsto (\tau_{12}, \tau_{23})$ is easy to check. To see that $r \mapsto R_3$, one should use the Fricke identity

$$\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\,\mathrm{tr}(B) \tag{5.5}$$

valid for $A, B \in \mathrm{SL}_2(R)$ for any ring R. To verify this identity, one uses the Cayley-Hamilton theorem to deduce that $A + A^{-1} = \mathrm{Tr}(A)$. Multiplying both sides by B and taking traces yields the identity (5.5).

The commutativity of the diagram follows from the lemma. Using the explicit form of the $\mathrm{Aut}(\Pi)$-action, the fact that $\mathrm{Aut}(\Pi)$ preserves the fibers amounts to the observation that for ring R and any homomorphism $\varphi : \Pi \to \mathrm{SL}_2(R)$, the following identities hold:

$$T(\tau_{12}(x,y,z)) = T(\tau_{23}(x,y,z)) = T(R_3(x,y,z)) = T(x,y,z),$$

where $\tau_{12}, \tau_{23}, R_3$ are as in Proposition 5.2.5. This is easily verified by hand. $\square$

Remark 5.2.6. The fact that $\mathrm{Aut}^+(\Pi)$ preserves the fibers of τ also follows from the local constancy of the Higman invariant (see Proposition 2.3.2). The fact that $\mathrm{Aut}(\Pi)$ preserves the fibers can be checked by noting that any representative of the non-trivial coset in $\mathrm{Aut}(\Pi)/\mathrm{Aut}^+(\Pi)$ sends the Higman invariant to its inverse, and traces in SL_2 are invariant under inversion.

Definition 5.2.7. Let T be a scheme, $n \geq 1$ an integer, and G a group. A representation $\varphi : G \to \mathrm{GL}_n(T) = \mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$ is *absolutely irreducible* if the induced algebra homomorphism $\Gamma(T, \mathcal{O}_T)[G] \to M_n(\Gamma(T, \mathcal{O}_T))$ is surjective. A subgroup $G \subset \mathrm{GL}_n(T)$ is *absolutely irreducible* if the inclusion $G \hookrightarrow \mathrm{GL}_n(T)$ is an absolutely irreducible representation.

When $T = \operatorname{Spec} k$ with k a field, then this notion of absolute irreducibility is the same as the non-existence of non-trivial G-invariant subspaces of $\overline{k}^n$, where $\overline{k}$ denotes the algebraic closure [Lan02, §XVII, Cor. 3.4].

Definition 5.2.8. Let R be any ring. Let $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})^{\mathrm{ai}} \subset \mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})$ be the subfunctor corresponding to the absolutely irreducible representations. By [Nak00, §3], this is represented by an open subscheme of $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})$. Accordingly, let

$$X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}} := \mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})^{\mathrm{ai}} /\!\!/ \mathrm{GL}_2 .$$

As usual, if $R = \mathbb{Z}$, then we will simply write $X^{\mathrm{ai}}_{\mathrm{SL}_2} := X^{\mathrm{ai}}_{\mathrm{SL}_{2,\mathbb{Z}}}$.

Lemma 5.2.9. *Let R be a ring. Let $\varphi : \Pi \to \mathrm{SL}_2(R)$ be a representation. Let $(A, B, C) := (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab))$. Then φ is absolutely irreducible if and only if*

$$A^2 + B^2 + C^2 - ABC - 4 \in R^\times .$$

In particular, we have $X^{\mathrm{ai}}_{\mathrm{SL}_2} = X_{\mathrm{SL}_2} - \tau^{-1}(2)$.

Proof. The first statement is [BH95, Prop. 4.1]. The second statement follows from the description of τ in Proposition 5.2.5. □

We have the following "moduli interpretation" for $X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}}(\mathbb{F}_q)$.

Theorem 5.2.10. *Let Π be a free group on the generators a, b. Let $q = p^r$ be a prime power. Let $\mathrm{GL}_2(\mathbb{F}_q)$ act on the set $\mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))$ by conjugation. The map*

$$\begin{aligned} \mathrm{Tr} : \mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q)) &\longrightarrow \mathbb{F}_q^3 \\ \varphi &\mapsto (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab)) \end{aligned}$$

is surjective. *Moreover, the following maps induced by* Tr *are* bijections:

$$\mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))^{\mathrm{ai}} / \mathrm{GL}_2(\mathbb{F}_q) \xrightarrow{\alpha} X^{\mathrm{ai}}_{\mathrm{SL}_2}(\mathbb{F}_q) \xrightarrow{\mathrm{Tr}} \mathbb{F}_q^3 - T^{-1}(2). \tag{5.6}$$

The surjectivity of Tr is [Mac69, Th. 1]. We will give two proofs of the bijectivity statement. The first is elementary and amounts to a classical result of Macbeath [Mac69] using explicit calculations in $\mathrm{SL}_2(\overline{\mathbb{F}_p})$. The second is significantly more general and uses results of Nakamoto [Nak00] on character varieties which, in particular, shows that the map $\mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})^{\mathrm{ai}} \to X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}}$ is a universal geometric quotient.

Proof 1. The second map $\mathrm{Tr} : X^{\mathrm{ai}}_{\mathrm{SL}_2}(\mathbb{F}_q) \longrightarrow \mathbb{F}_q^3 - T^{-1}(2)$ is already a bijection by Proposition 5.2.5. Thus it suffices to show that the composition is bijective. This amounts to a classical result of Macbeath. For a homomorphism $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$, let $(A, B, C) = (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab))$ and let

$$Q_\varphi(x, y, z) = Q_{A,B,C}(x, y, z) := x^2 + y^2 + z^2 + Ayz + Bxz + Cxy.$$

We say that φ is non-singular if the projective conic defined by Q_φ is smooth (equivalently geometrically integral). Similarly, given any triple $(A, B, C) \in \mathbb{F}_q^3$, we say that it is non-singular if the form $Q_{A,B,C}$ defines a smooth projective conic. We claim that φ is non-singular if and only if it is absolutely irreducible. Indeed, the discriminant of $Q_\varphi = Q_{A,B,C}$ is

$$\mathrm{disc}(Q_\varphi) = \mathrm{disc}(Q_{A,B,C}) = -(A^2 + B^2 + C^2 - ABC - 4),$$

and the associated conic fails to be smooth if and only if $\mathrm{disc}(Q_\varphi) = 0$,[32] or equivalently, when $T(A, B, C) = 2$, where T is as in Proposition 5.2.5. By Lemma 5.2.9, this is equivalent to φ not being absolutely irreducible, so the non-singular representations are precisely the absolutely irreducible representations, and the non-singular triples are precisely those in $\mathbb{F}_q^3 - T^{-1}(2)$.

In [Mac69, Th. 3], Macbeath shows that every non-singular triple in $\mathbb{F}_q^3$ is the trace of a non-singular φ, and conversely any two non-singular representations φ, φ' are conjugate by some $P \in \mathrm{SL}_2(\overline{\mathbb{F}_q})$. Since φ, φ' are absolutely irreducible, the associated algebra homomorphisms $\widehat{\varphi}, \widehat{\varphi'} : \mathbb{F}_q[\Pi] \to M_2(\mathbb{F}_q)$ are surjective, and hence such a P must normalize $M_2(\mathbb{F}_q)$, so it must normalize $\mathrm{SL}_2(\mathbb{F}_q)$. Finally, it follows from the description of the normalizer $N_{\mathrm{GL}_2(\overline{\mathbb{F}_q})}(\mathrm{SL}_2(\mathbb{F}_q))$ in Proposition 6.3.1 that the action factors through the action of $\mathrm{GL}_2(\mathbb{F}_q)$, so Macbeath's result establishes the desired bijectivity. ☐

Proof 2. Since Tr is a bijection (Theorem 5.2.1, Proposition 5.2.5), it remains to prove the bijectivity of α. It follows from [Nak00, Cors. 2.13 and 6.8] that for any ring R, the map

$$\xi : \mathrm{Hom}(\Pi, \mathrm{SL}_{2,R})^{\mathrm{ai}} \to X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}}$$

is a *universal geometric quotient* by $\mathrm{GL}_{2,R}$ in the sense of geometric invariant theory [MFK94, Def. 0.6]. Thus for algebraically closed fields Ω over R, ξ induces a bijection

$$\mathrm{Hom}(\Pi, \mathrm{SL}_2(\Omega))^{\mathrm{ai}} / \mathrm{GL}_2(\Omega) \xrightarrow{\sim} X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}}(\Omega).$$

Setting $\Omega = \overline{\mathbb{F}_q}$, Proposition 6.3.1 implies (arguing as in the first proof) that two absolutely irreducible representations $\varphi, \varphi' : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$, are conjugate in $\mathrm{GL}_2(\overline{\mathbb{F}_q})$ if and only if they are conjugate in $\mathrm{GL}_2(\mathbb{F}_q)$. This implies the injectivity of α. Since the center of $\mathrm{GL}_{2,R}$ acts trivially, ξ is also a universal geometric quotient by $\mathrm{PGL}_{2,R}$. Since our representations are absolutely irreducible, the

[32]The definition of the discriminant and its association with the smoothness of the associated conic is classical in characteristic $p \neq 2$, but in characteristic 2 some care is needed. A modern treatment describing the discriminant for an arbitrary projective hypersurface over arbitrary fields is given in Demazure [Dem12]. The definition of discriminant is [Dem12, Def. 4] (also see [Dem12, Ex. 6]), and its association with smoothness is [Dem12, Prop. 12]. For a ternary quadratic form $q(x, y, z)$ over a field k, to calculate its discriminant one should first lift q to a quadratic form $\tilde{q}$ in characteristic 0 (e.g., over a Cohen ring of k), compute its discriminant there, defined as one half of the determinant of the Gram matrix of the bilinear form $b(v, w) = \tilde{q}(v + w) - \tilde{q}(v) - \tilde{q}(w)$, and then take its image in k.

$\mathrm{PGL}_{2,R}$-action is *free* [Nak00, Cor. 6.5], so ξ is moreover a principal $\mathrm{PGL}_{2,R}$-bundle. In particular, for any field k and map $x : \operatorname{Spec} k \to X^{\mathrm{ai}}_{\mathrm{SL}_{2,R}}$, the restriction $x^*\xi : x^* \operatorname{Hom}(\Pi, \mathrm{SL}_{2,R})^{\mathrm{ai}} \to \operatorname{Spec} k$ is a principal $\mathrm{PGL}_{2,k}$ bundle. When $k = \mathbb{F}_q$, by Lang's theorem [Poo17, Th. 5.12.19], principal $\mathrm{PGL}_{2,\mathbb{F}_q}$-bundles over $\operatorname{Spec} \mathbb{F}_q$ are trivial, and hence α is surjective.[33] $\square$

Remark 5.2.11. The methods used in the second proof also hold for character varieties of absolutely irreducible representations of arbitrary groups in arbitrary dimension. More precisely, if Π temporarily denotes an *arbitrary* group, $n \geq 1$ an integer, R any ring, and $X^{\mathrm{ai}}_{\Pi,\mathrm{GL}_n}$ the quotient of the representation variety of absolutely irreducible representations $\operatorname{Hom}(\Pi, \mathrm{GL}_{n,R})^{\mathrm{ai}}$ by $\mathrm{GL}_{n,R}$, then by [Nak00, Cor. 6.8], the quotient map $\operatorname{Hom}(\Pi, \mathrm{GL}_{n,R})^{\mathrm{ai}} \to X^{\mathrm{ai}}_{\Pi,\mathrm{GL}_n}$ is a universal geometric quotient and a principal $\mathrm{PGL}_{n,R}$-bundle, so the argument used in the second proof also gives a bijection $\operatorname{Hom}(\Pi, \mathrm{GL}_n(\mathbb{F}_q))^{\mathrm{ai}} / \mathrm{GL}_2(\mathbb{F}_q) \xrightarrow{\sim} X^{\mathrm{ai}}_{\Pi,\mathrm{GL}_n}(\mathbb{F}_q)$. Combined with Theorem 3.5.1, restricting to the subsets with image of a particular type, this bijection can potentially be used to establish congruences on the $\operatorname{Aut}(\Pi)$-orbits on $\mathbb{F}_q$ points of more general character varieties.

By Theorem 5.2.10, elements of $X_{\mathrm{SL}_2}(\mathbb{F}_q)$ do not quite correspond to $\mathrm{SL}_2(\mathbb{F}_q)$-structures, but rather $\mathrm{GL}_2(\mathbb{F}_q)$-equivalence classes of such structures.

Definition 5.2.12. For a prime power q, let $i : \mathrm{GL}_2(\mathbb{F}_q) \to \operatorname{Aut}(\mathrm{SL}_2(\mathbb{F}_q))$ be the map which sends $A \in \mathrm{GL}_2(\mathbb{F}_q)$ to the corresponding action by conjugation on $\mathrm{SL}_2(\mathbb{F}_q)$. Let $D(q)$ be the image of $\mathrm{GL}_2(\mathbb{F}_q)$ in $\operatorname{Out}(\mathrm{SL}_2(\mathbb{F}_q))$. We call $D(q)$ the group of "diagonal" outer automorphisms of $\mathrm{SL}_2(\mathbb{F}_q)$.

For a general subgroup $G \leq \mathrm{SL}_2(\mathbb{F}_q)$, let $D(q, G)$ denote the image of $i : N_{\mathrm{GL}_2(\mathbb{F}_q)}(G) \to \operatorname{Out}(G)$, where i is given by the conjugation action. Thus if $G = \mathrm{SL}_2(\mathbb{F}_q)$, then $D(q, G) = D(q)$.

PROPOSITION 5.2.13. *Let $q = p^r$ be a prime power. If q is even, $D(q)$ is trivial. If q is odd, $D(q)$ has order 2, and the non-trivial element is represented*

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[33] Alternatively, one could argue surjectivity as follows. Let k^s denote a separable closure of k. The exact sequence

$$1 \longrightarrow \mathbb{G}_m(k^s) \longrightarrow \mathrm{GL}_2(k^s) \longrightarrow \mathrm{PGL}_2(k^s) \longrightarrow 1$$

induces a longer exact sequence of Galois cohomology sets [Ser02, I,§5.7, Prop. 43]

$$\cdots \longrightarrow H^1(k, \mathrm{GL}_2(k^s)) \longrightarrow H^1(k, \mathrm{PGL}_2(k^s)) \longrightarrow H^2(k, \mathbb{G}_m(k^s)) \longrightarrow \cdots .$$

Since PGL_2 is smooth, every principal PGL_2-bundle over k^s is trivial, and hence $H^1(k, \mathrm{PGL}_2(k^s))$ classifies principal PGL_2-bundles over k [Poo17, §5.12.4]. On the other hand, the first term of the sequence vanishes by Hilbert's Theorem 90 [Poo17, Prop. 1.3.15], and the last is the Brauer group, which vanishes for finite fields (or any field of cohomological dimension ≤ 1). In particular, if $k = \mathbb{F}_q$, then $x^*\xi$ is a trivial $\mathrm{PGL}_{2,k}$-bundle, and hence α is surjective.

by any matrix $A \in \mathrm{GL}_2(\mathbb{F}_q)$ *with non-square determinant. When* $q = p$ *is a prime,* $D(p) = \mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$.

Proof. The fact that $D(p) = \mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$ follows from [Ste16, Th. 30], noting that for $\mathrm{SL}_2(\mathbb{F}_p)$, there are no graph or field automorphisms. The rest is clear. □

Definition 5.2.14. For a prime power q, we say that a subgroup $G \leq \mathrm{SL}_2(\mathbb{F}_q)$ is *of dihedral type* if its image in $\mathrm{PSL}_2(\mathbb{F}_q) := \mathrm{SL}_2(\mathbb{F}_q)/\{\pm I\}$ is either trivial or isomorphic to the dihedral group of order $2n$ for some $n \geq 1$. For a representation $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$, we say that it is of *dihedral type* if its image is a subgroup of dihedral type.

For the results in the next section we will eventually need to exclude the representations of dihedral type. Next we make some preliminary observations about such representations.

Definition 5.2.15. Let R be a ring, and let $t \in R$ be an element corresponding to a map $t : \operatorname{Spec} R \to \mathbb{A}^1$. Let

$$X_{\mathrm{SL}_2,t} := \tau^{-1}(t) := X_{\mathrm{SL}_2} \times_{\tau,\mathbb{A}^1,t} \operatorname{Spec} R$$

be the fiber corresponding to representations with $\operatorname{tr}\varphi([a,b]) = t$. Via $\mathrm{Tr} : X_{\mathrm{SL}_2,R} \xrightarrow{\sim} \mathbb{A}^3_R$, $X_{\mathrm{SL}_2,t} = \tau^{-1}(t)$ is identified with the affine surface $T^{-1}(t) \subset \mathbb{A}^3_R$:

$$T^{-1}(t) : x^2 + y^2 + z^2 - xyz - 2 = t.$$

In particular, when $t = -2 \in \mathbb{Z}$, this is the Markoff surface $\mathbb{X} \subset \mathbb{A}^3_{\mathbb{Z}}$. By Theorem 5.2.10, for $t \in \mathbb{F}_q$, every $\mathbb{F}_q$ point of $T^{-1}(t)$ is the image under Tr of a representation $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$.

- Let $X_t(q)$ denote the set of $\mathbb{F}_q$-points of the affine surface $T^{-1}(t) \subset \mathbb{A}^3_{\mathbb{F}_q}$.
- Let $X^*_t(q) \subset X_t(q)$ denote the subset corresponding to absolutely irreducible representations which are not of dihedral type (Definition 5.2.14).
- Let $X^\circ_t(q) \subset X_t(q)$ denote the subset corresponding to *surjective* representations $\Pi \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_q)$.
- Let $X(q) := \bigsqcup_{t\in\mathbb{F}_q} X_t(q)$, let $X^*(q) := \bigsqcup_{t\in\mathbb{F}_q} X^*_t(q)$, and let $X^\circ(q) := \bigsqcup_{t\in\mathbb{F}_q} X^\circ_t(q)$.

PROPOSITION 5.2.16. *The following are true:*

(a) *For all prime powers* q, $X^*_2(q)$ *and* $X^\circ_2(q)$ *are empty.*

(b) *For* $q \geq 3$, $X^\circ(q) \subset X^*(q)$.

(c) *For* $q = 2$, $X^*(2)$ *is empty whereas* $X^\circ(2)$ *is not.*

(d) *For any prime power* q, *if* $t \neq 2 \in \mathbb{F}_q$, *then*

$$X^*_t(q) = \{(x,y,z) \in X_t(q) \mid \textit{as least two of } \{x,y,z\} \textit{ are non-zero in } \mathbb{F}_q\}.$$

Proof. Part (a) follows immediately from Lemma 5.2.9. Part (b) follows from the fact that $\mathrm{PSL}_2(\mathbb{F}_q)$ is not dihedral for $q \geq 3$. For $q = 2$, $X^*(q)$ is empty since every subgroup of $\mathrm{SL}_2(\mathbb{F}_2) = \mathrm{PSL}_2(\mathbb{F}_2)$ is either dihedral or cyclic. On the other hand, there exist surjections $\Pi \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_2)$, so $X^\circ(2)$ is non-empty. This proves (c).

Finally we prove (d). Let $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$ be an absolutely irreducible representation with $A := \varphi(a), B := \varphi(b)$. Recall that φ is absolutely irreducible if and only if $\operatorname{tr}\varphi([a, b]) = 2$ (Lemma 5.2.9). Thus, $X_2^*(q)$ is empty, and it remains to show that φ is of dihedral type if and only if two of $\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB$ are equal to 0. Note that a group is dihedral if and only if it is generated by two elements g, h with $g^2 = h^2 = 1$, and that a matrix $A \in \mathrm{SL}_2(\mathbb{F}_q)$ maps to an element of order 2 in $\mathrm{PSL}_2(\mathbb{F}_q)$ if and only if $\operatorname{tr}(A) = 0$. Thus if two of $\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB$ are equal to zero, then φ must be of dihedral type. Conversely, if φ is of dihedral type, let $\overline{A}, \overline{B} \in \mathrm{PSL}_2(\mathbb{F}_q)$ be the images of A, B. Then $\overline{A}, \overline{B}$ must satisfy $\overline{A}^2 = \overline{B}^2 = 1$, and up to switching A, B, there are three cases:

- $|\overline{A}| = |\overline{B}| = 1$. Then $\varphi(\Pi) = \langle A, B\rangle \leq Z(\mathrm{SL}_2(\mathbb{F}_q))$, so φ is not absolutely irreducible, so this case cannot occur.
- $|\overline{A}| = 1$ and $|\overline{B}| = 2$. Then $A = \pm I$ and $\operatorname{tr}(B) = \operatorname{tr}(AB) = 0$. In this case $\operatorname{Tr}(\varphi) = (\pm 2, 0, 0)$ (though this case is actually forbidden since $\langle A, B\rangle$ is cyclic so φ cannot be absolutely irreducible in this case).
- $|\overline{A}| = |\overline{B}| = 2$. Then $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. In this case $\operatorname{Tr}(\varphi) = (0, 0, \operatorname{tr}(AB))$.

Thus a representation of dihedral type must have at least two of $\operatorname{tr}\varphi(a)$, $\operatorname{tr}\varphi(b)$, $\operatorname{tr}\varphi(ab)$ equal to 0, as desired. □

For us, an important special case is when $q = p$ and $t = -2$,

Proposition 5.2.17. *The following are true*:

(a) *For all primes* $p \geq 3$, $X^*_{-2}(p) = X^\circ_{-2}(p)$ *consists of the* $\mathbb{F}_p$*-points of the surface*

$$\mathbb{X} : x^2 + y^2 + z^2 - xyz = 0$$

other than $(0, 0, 0)$. *In other words, for* $p \geq 3$, *we have* $X^*_{-2}(p) = X^\circ_{-2}(p) = X_{-2}(p) - \{(0, 0, 0)\} = \mathbb{X}^*(p)$.

(b) *For* $p = 2$, $X^\circ_{-2}(2) = X^*_{-2}(2)$ *is empty but*

$$X_{-2}(2) - \{(0, 0, 0)\} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.$$

Proof. Note that in $X_{-2}(p)$, if two of the coordinates are zero, then the third must also be zero. The fact that $X^\circ_{-2}(p) = X_{-2}(p) - \{(0, 0, 0)\}$ for $p \geq 3$ follows from the analysis of [MW13, §11]. This also implies $X^*_{-2}(p) = X_{-2}(p) - \{(0, 0, 0)\}$ for $p \geq 3$ by Proposition 5.2.16(b). For $p = 2$, $-2 = 2$, so the statement in this case follows from Proposition 5.2.16(a). □

Remark 5.2.18. For general prime powers q and traces $t \in \mathbb{F}_q$, the discussion in [MW13, §11] gives a complete (albeit more complicated) description of the subsets $X_t^\circ(q) \subset X(q)$.

Definition 5.2.19. Given a subgroup $G \le \mathrm{SL}_2(\mathbb{F}_q)$, we may speak of the trace of elements $g \in G$. For any $t \in \mathbb{F}_q$, let $\mathcal{A}dm(G)_t \subset \mathcal{A}dm(G)_{\overline{\mathbb{Q}}}$ denote the open and closed substack classifying G-covers whose Higman invariant has trace t. We have analogous notions of $Adm(G)_t, \mathcal{M}(G)_t, M(G)_t$.

Here we summarize the relation between the $\mathrm{Aut}(\Pi)$-action on $X^*(q)$ and the geometry of the moduli stacks $\mathcal{M}(G)_{\overline{\mathbb{Q}}}/D(q,G)$:

PROPOSITION 5.2.20. *Let $p \ge 3$ be a prime. Let E, Π, a, b, x_E be as in Situation 2.5.14. Let $G \le \mathrm{SL}_2(\mathbb{F}_q)$ be the image of an absolutely irreducible representation $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$, and let $D(q,G)$ be as in Definition 5.2.12. Let*

$$\mathfrak{f} : \mathcal{M}(G)_{\overline{\mathbb{Q}}}/D(q,G) \longrightarrow \mathcal{M}(1)$$

be the forgetful map. We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{f}^{-1}(x_E) \arrow[r, "\cong"] \arrow[drr, bend right, "\text{trace invariant}"'] & \mathrm{Epi}^{\mathrm{ext}}(\Pi,G)/D(q,G) \arrow[r, hook] \arrow[dr, "\mathrm{tr}\,\varphi({[a,b]})"'] & X^*(q) \subset \mathbb{F}_q^3 \arrow[d, "x^2+y^2+z^2-xyz-2"] \\
& & \mathbb{F}_q - \{2\},
\end{tikzcd} \tag{5.7}$$

where the map "trace invariant" is the trace invariant of any G-structure in the $D(q,G)$-orbit. The automorphisms $\gamma_0, \gamma_{1728}, \gamma_\infty, \gamma_{-I} \in \mathrm{Aut}^+(\Pi)$ and their induced actions on $X(q) := \sqcup_{t\in\mathbb{F}_q} X_t(q) = \mathbb{F}_q^3$ are given as follows:

$$\begin{array}{rcl c rcl}
\gamma_0 : (a,b) & \mapsto & (ab^{-1}, a), & & (x,y,z) & \mapsto & (xy - z, x, x^2y - xz - y), \\
\gamma_{1728} : (a,b) & \mapsto & (b^{-1}, a), & \xrightarrow{\mathrm{Tr}_*} & (x,y,z) & \mapsto & (y, x, xy - z), \\
\gamma_\infty : (a,b) & \mapsto & (a, ab), & & \mathrm{rot}_1 : (x,y,z) & \mapsto & (x, z, xz - y), \\
\gamma_{-I} : (a,b) & \mapsto & (a^{-1}, b^{-1}), & & (x,y,z) & \mapsto & (x,y,z).
\end{array}$$

By Galois theory, to every $\mathrm{Aut}^+(\Pi)$-orbit on $X(q)$ is associated a connected component of $\mathcal{M}(G)_{\overline{\mathbb{Q}}}/D(q,G)$ for some G as above. If $P \in X(q)$ corresponds to a geometric point $x_P \in \mathcal{M}(G)_{\overline{\mathbb{Q}}}/D(q,G)$, then the connected component $\mathcal{M}(x_P) \subset \mathcal{M}(G)_{\overline{\mathbb{Q}}}/D(q,G)$ containing P has degree over $\mathcal{M}(1)_{\overline{\mathbb{Q}}}$ equal to the size of the $\mathrm{Aut}^+(\Pi)$-orbit of P. Let $M(x_P)$ denote the coarse scheme, let $\overline{M(x_P)}$ denote its smooth compactification, and let

$$f : \overline{M(x_P)} \to \overline{M(1)}_{\overline{\mathbb{Q}}}$$

denote the forgetful map. If we identify $\overline{M(1)}_{\overline{\mathbb{Q}}}$ with the projective line with coordinate j, then f is a branched covering of smooth proper curves over $\overline{\mathbb{Q}}$, étale over the complement of the points $j = 0, 1728, \infty$. If $j(E) \ne 0, 1728$, then *the bijection in* (5.7) *induces a bijection*

$$f^{-1}(x_E) \xrightarrow{\sim} \left(\mathrm{Aut}^+(\Pi)\cdot P\right)/\langle\gamma_{-I}\rangle = \mathrm{Aut}^+(\Pi)\cdot P.$$

For $j = 0, 1728, \infty$, *there are bijections*

$$f^{-1}(0) = \left(\mathrm{Aut}^+(\Pi)\cdot P\right)/\langle\gamma_0\rangle, \quad f^{-1}(1728) = \left(\mathrm{Aut}^+(\Pi)\cdot P\right)/\langle\gamma_{1728}\rangle, \text{ and}$$
$$f^{-1}(\infty) = \left(\mathrm{Aut}^+(\Pi)\cdot P\right)/\langle\gamma_\infty\rangle$$

such that the ramification index of a given point in $f^{-1}(0)$ (*resp.* $f^{-1}(1728)$, $f^{-1}(\infty)$) *is equal to the size of the corresponding orbit under* γ_0 (*resp.* $\gamma_{1728}, \gamma_\infty$) *in* $\left(\mathrm{Aut}^+(\Pi)\cdot P\right)$.

Proof. The properties of the diagram (5.7) follows from Theorem 5.2.10, Proposition 5.2.5, and the discussion in Situation 2.5.14. The formulas for the automorphisms of $X(q) := \bigsqcup_{t\in\mathbb{F}_q} X_t(q) = \mathbb{F}_q^3$ induced by $\gamma_0, \gamma_{1728}, \gamma_\infty$ can be verified using Proposition 5.2.5, noting that

$$\gamma_0 = s\circ r\circ s\circ t\circ r\circ s, \ \ \gamma_{1728} = s\circ r, \ \ \gamma_\infty = t\circ r, \ \ \gamma_{-I} = \gamma_{1728}^2.$$

Here we must remember that $\mathrm{Tr}_* : \mathrm{Aut}(\Pi) \to \mathrm{Aut}(X(q))$ is an *anti-homomorphism*. Since $\gamma_{-I} = \gamma_{1728}^2$ acts trivially on $X^\circ(p)$, the statement about ramification indices follows from [BBCL22, Prop. 2.2.3]. ☐

Remark 5.2.21. The automorphisms $\gamma_0, \gamma_{1728}, \gamma_\infty, \gamma_{-I}$ are perhaps more familiar in terms of the corresponding matrices they determine in $\mathrm{SL}_2(\mathbb{Z})$. Namely, under the map $\Pi \to \mathbb{Z}^2$, sending a, b to the canonical basis of $\mathbb{Z}^2$, we find that $\gamma_0, \gamma_{1728}, \gamma_\infty, \gamma_{-I}$ correspond to the matrices

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

respectively. Viewing the matrices as fractional linear transformations of the upper half plane $\mathcal{H}$, the j-function takes the values $0, 1728, \infty$ respectively on the fixed point sets of the first three matrices. The final matrix acts trivially on $\mathcal{H}$ and corresponds to the hyperelliptic involution.

Remark 5.2.22. By Proposition 5.2.17(a), $X^*_{-2}(p) = X^\circ_{-2}(p)$ (for $p \geq 3$) is precisely the set of non-zero $\mathbb{F}_p$-points of $x^2+y^2+z^2-xyz = 0$. Thus it follows from the combinatorial analysis in the previous section that every $\mathrm{Out}^+(\Pi)$-orbit on this set has cardinality divisible by p (see Corollary 4.12.5(a)). This proves Theorem 1.2.5 given in the introduction, and hence resolves the conjecture of Bourgain, Gamburd, and Sarnak for all but finitely many primes. In the following sections we will give a different proof of this fact using the explicit form of the $\mathrm{Aut}^+(\Pi)$-action on $X(q) = \sqcup_{t\in\mathbb{F}_q} X_t(q)$; this will yield additional congruences not directly implied by Corollary 4.12.5.

5.3. *Automorphism groups of* $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\overline{\mathbb{Q}}}$. In this section we use the formalism developed in Section 5.2 to describe the vertical automorphism groups of geometric points of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\overline{\mathbb{Q}}}$. The main result is that for any $q \geq 3$, the map $\overline{\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_q)}_{\overline{\mathbb{Q}}} \to \overline{\mathcal{M}(1)}_{\overline{\mathbb{Q}}}$ is representable. By Proposition 2.5.10(b), the restriction to the preimage above $\mathcal{M}(1)_{\overline{\mathbb{Q}}}$ is just the finite étale map $\mathcal{M}(G)_{\overline{\mathbb{Q}}} \to \mathcal{M}(1)_{\overline{\mathbb{Q}}}$ (Theorem 2.5.2(1)), so there the representability is a consequence of finiteness. It remains to show that the vertical automorphism groups remain trivial at the cusps. For this, consider the "Dehn twist" $\gamma_\infty \in \mathrm{Aut}(\Pi)$ given by

$$\gamma_\infty = t \circ r : (a, b) \mapsto (a, ab),$$

which induces the "rotation" (using the terminology of [BGS16a])

$$\mathrm{rot}_1 := \mathrm{Tr}_*(t \circ r) = \mathrm{Tr}_*(r) \circ \mathrm{Tr}_*(t) = R_3 \circ \tau_{23} : (x, y, z) \mapsto (x, z, xz - y).$$

By Theorem 4.8.4, the cusps of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_{\overline{\mathbb{Q}}}$ are classified by the set $\mathbb{I}(\mathrm{SL}_2(\mathbb{F}_q))$. Let Π, a, b be as in Situation 2.5.14. Then the map

$$\begin{aligned} \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_q)) &\longrightarrow \mathbb{I}(\mathrm{SL}_2(\mathbb{F}_q)), \\ \varphi &\mapsto [\![\varphi(a), \varphi(b)]\!] \end{aligned}$$

induces a bijection $\mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))/\langle\gamma_\infty\rangle \xrightarrow{\sim} \mathbb{I}(\mathrm{SL}_2(\mathbb{F}_q))$. It follows from Theorem 4.10.3(a) that the following are equivalent:

- For every $\varphi \in \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))$, the orbit $\{\varphi \circ \gamma_\infty^i \mid i \in \mathbb{Z}\}$ has size $|\varphi(a)|$.
- The vertical automorphism groups of geometric points of $\mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))$ are reduced to $\{\pm I\} = Z(\mathrm{SL}_2(\mathbb{F}_q))$.

Using the moduli interpretation of the points $X_{\mathrm{SL}_2}(\mathbb{F}_q)$ (Theorem 5.2.10), we will show something even stronger:

LEMMA 5.3.1. *For any prime power $q \geq 3$, let $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$ be an absolutely irreducible representation with* $(\mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab)) \neq (0, 0)$. (*In particular, φ is not of dihedral type*; *see Proposition* 5.2.16.) *Then the γ_∞-orbit of φ viewed as an element of* $\mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))^{\mathrm{ai}}/D(q)$ *has size* $|\varphi(a)|$.

The proof of the lemma will be given below. Recall that for $t \in \mathbb{F}_q$, $n_q(t)$ is the order of any non-central element of $\mathrm{SL}_2(\mathbb{F}_q)$ of trace t (Definition 5.1.4). The lemma implies

THEOREM 5.3.2. *Let $q \geq 3$ be a prime power. Let $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$ be an absolutely irreducible representation which is not of dihedral type*. (*Equivalently, $\varphi(\Pi)$ is not a subgroup of dihedral type and* $\mathrm{tr}\,\varphi([a, b]) \neq 2$.) *Then $G := \varphi(\Pi)$ contains $Z(\mathrm{SL}_2(\mathbb{F}_q))$ and we have*

(a) *the vertical automorphism groups of geometric points of $\mathcal{A}dm(G)_{\overline{\mathbb{Q}}}$ are all reduced to $Z(\mathrm{SL}_2(\mathbb{F}_q)) = \{\pm I\}$*;

(b) *the forgetful map* $\mathfrak{f} : \overline{\mathcal{M}(G)_{\overline{\mathbb{Q}}}} \longrightarrow \overline{\mathcal{M}(1)_{\overline{\mathbb{Q}}}}$ *is representable.*

Let $t \neq 2 \in \mathbb{F}_q$, *and let* $\mathcal{X} \subset \mathcal{A}dm(G)_t$ *be a component with universal family* $\pi : \mathcal{C} \to \mathcal{E}$ *and reduced ramification divisor* $\mathcal{R}_\pi$. *The vertical automorphism groups of geometric points of* $\mathcal{R}_\pi$ *are as follows*:

(c1) *If* q *is even, then for any geometric point* $x \in \mathcal{R}_\pi$, $\mathrm{Aut}^v(x)$ *is trivial.*

(c2) *If* q *is odd and* $t = -2$, *then for any geometric point* $x \in \mathcal{R}_\pi$, $\mathrm{Aut}^v(x)$ *has order* 2.

(c3) *If* q *is odd,* $t \neq -2$ *and* $n_q(t)$ *is even, then for any geometric point* $x \in \mathcal{R}_\pi$, $\mathrm{Aut}^v(x)$ *has order* 2.

(c4) *If* q *is odd,* $t \neq -2$ *and* $n_q(t)$ *is odd, then for any geometric point* $x \in \mathcal{R}_\pi$, $\mathrm{Aut}^v(x)$ *is trivial.*

Proof of Theorem 5.3.2. That $G = \varphi(\Pi)$ must contain the $Z(\mathrm{SL}_2(\mathbb{F}_q))$ follows from Macbeath's classification of the 2-generated subgroups of $\mathrm{SL}_2(\mathbb{F}_q)$ [Mac69, §4] (also see Proposition 6.4.1 in the appendix). Parts (a) and (b) follow immediately from Lemma 5.3.1 and the preceding discussion. It remains to address (c1)–(c4). Suppose $\mathcal{X}$ classifies covers with Higman invariant $\mathfrak{c}$. Let $c \in \mathfrak{c}$ be a representative. Using Theorem 4.10.3 and Lemma 5.3.1, we find $\mathrm{Aut}^v(x) = Z(\mathrm{SL}_2(\mathbb{F}_q)) \cap \langle c\rangle$. If q is even, then $Z(\mathrm{SL}_2(\mathbb{F}_q)) = 1$ so we are done. Now assume q is odd. If $t = -2$, then c is conjugate to $\left[\begin{smallmatrix} -1 & u \\ 0 & -1 \end{smallmatrix}\right]$ for some $u \in \mathbb{F}_q^\times$, so in this case $c^p = -I$ so $\mathrm{Aut}^v(x)$ has order 2. If $t \neq -2$, then c is diagonalizable over $\overline{\mathbb{F}_q}$, so $\langle c\rangle$ has non-trivial intersection with $\{\pm I\}$ if and only if $|c| = n_q(t)$ is even. ☐

Finally we prove Lemma 5.3.1. Along the way we will also count the number of rot_1 orbits on $X^\circ_{-2}(p)$ for $p \geq 3$.

Proof of Lemma 5.3.1. Since $\mathrm{Tr}_*(\gamma_\infty) = \mathrm{rot}_1$, we must analyze the action of rot_1 on the $X_t(q)$ for various $t \in \mathbb{F}_q$. For any $a \in \mathbb{F}_q, t \in \mathbb{F}_q$, the action of rot_1 visibly preserves the conics

$$C_1(a)_t := X_t(q)_{x=a} = \{(x, y, z) \in \mathbb{F}_q^3 \mid x = a \text{ and }$$
$$y^2 + z^2 - ayz + (a^2 - 2 - t) = 0\} \subset X_t(q),$$

where q is understood. Since rot_1 is induced by an action of $\gamma_\infty \in \mathrm{Aut}(\Pi)$, it also preserves the subset

$$C_1(a)^*_t := X^*_t(q) \cap C_1(a)_t$$

corresponding to absolutely irreducible representations not of dihedral type, as well as the subset

$$C_1(a)^\circ_t := X^\circ_t(q) \cap C_1(a)_t$$

corresponding to surjective representations. Note that for $q \geq 3$, $\mathrm{PSL}_2(\mathbb{F}_q)$ is not dihedral, so $C_1(a)^\circ_t \subset C_1(a)^*_t$ for $q \geq 3$.

Here we analyze the action of rot_1 on these conics, following [BGS16a] and [MP18]. Homogenizing the equation $y^2+z^2-ayz+(a^2-2-t)=0$, by Chevalley-Warning we find that if $a^2 \neq t+2 \in \mathbb{F}_q$, then $C_1(a)_t$ is non-empty. The discriminant of the corresponding ternary quadratic form is $(4-a^2)(a^2-2-t)$ and hence we find that $C_1(a)_t$ is a degenerate conic if and only if $a^2=4$ or $a^2=t+2$ in $\mathbb{F}_q$. Thus we will consider the values $a=\pm 2, \pm\sqrt{t+2}$ separately, where $\sqrt{t+2} \in \overline{\mathbb{F}_q}$ is a square root of $t+2$. On each $C_1(a)_t$, rot_1 acts as the linear transformation on the ambient affine yz-plane given by

$$\mathrm{rot}_1|_{C_1(a)_t} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right].$$

Given a subset $Z \subset C_1(a)_t$, we say that rot_1 *acts freely on* Z if the cyclic group $\langle \left[\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right] \rangle$ acts freely on Z. We will analyze the action of rot_1 according to the behavior of a^2-4. We will show that for any $q \geq 3$ and $a, t \in \mathbb{F}_q$ with $t \neq 2$, rot_1 acts freely on $C_1(a)_t^*$, so its orbits all have the same size $n_q(a)$ (Definition 5.1.4). For later use, we will also count the number of rot_1-orbits in the case where $q=p\geq 3$ and $t=-2$ (in which case $C_1(a)_{-2}^* = C_1(a)_{-2}^\circ$ by Proposition 5.2.17).

Since $C_1(a)_2^* = C_1(a)_2^\circ = \emptyset$, in the following we only consider the case when $t \neq 2 \in \mathbb{F}_q$. In particular, the exceptional cases $a^2=4, a^2=t+2$ do not overlap. We will also restrict ourselves to the case $q \geq 3$ (equivalently, $\mathrm{PSL}_2(\mathbb{F}_q)$ is not dihedral).

(1) Suppose $a^2-4=0$ ($a=\pm 2$), $t\neq 2$, and $q\geq 3$ is odd. In this case we say a is parabolic.

If $a=2$, then $C_1(a)_t$ is given by $(y-z)^2=t-2$, which is non-empty if and only if $t-2 \in \mathbb{F}_q^\times$ is a square. When this is the case it is the disjoint union of the two lines $y-z=\pm\sqrt{t-2}$. On $\mathbb{F}_q^2$, rot_1 acts via $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right]$, which is conjugate to $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. It fixes pointwise the line $y=z$ in $\mathbb{F}_q^2$ and acts freely with order $n_q(a)=p$ everywhere else. Since $t\neq 2$, $C_1(a)_t$ does not contain any fixed points, so rot_1 acts on $C_1(a)_t$ freely with orbits of size $n_q(a)=p$. When $q=p\geq 3$ and $t=-2$, we obtain two orbits (of size p) on $C_1(2)_{-2}^*$ when $p\equiv 1 \mod 4$ and zero orbits if $p \equiv 3 \mod 4$.

If $a=-2$, then $C_1(a)_t$ is given by $(y+z)^2=t-2$, which is again non-empty if and only if $t-2\in\mathbb{F}_q^\times$ is a square. When $t-2$ is a square, $C_1(a)_t$ is again a disjoint union of the lines $y+z=\pm\sqrt{t-2}$. On $\mathbb{F}_q^\times$, rot_1 acts via $\left[\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$, which acts by $v \mapsto -v$ on the line $y=-z$ and acts freely everywhere else, alternating between the two lines. Since q is odd and $t\neq 2$, rot_1 acts on $C_1(a)_t$ freely with order $n_q(a)=2p$. When $q=p\geq 3$ and $t=-2$, we obtain one orbit (of size $2p$) on $C_1(-2)_{-2}^*$ when $p\equiv 1 \mod 4$ and zero orbits if $p\equiv 3 \mod 4$.

(2) Suppose $a^2-4\in\mathbb{F}_q^\times$ is a square, $t\neq 2$, and $q\geq 3$ is odd. In this case we say a is hyperbolic. Hyperbolic a's are in bijection with the set of polynomials

of the form $T^2 - aT + 1$ with distinct roots in $\mathbb{F}_q^\times$, and hence there are $\frac{q-3}{2}$ hyperbolic a's.

In this case rot_1 acts via $\left[\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right]$, which is diagonalizable with distinct eigenvalues ω, ω^{-1} with $\omega = \frac{a \pm \sqrt{a^2-4}}{2}$ and $a = \omega + \omega^{-1}$. This matrix acts freely on $\mathbb{F}_q^2 - \{(0,0)\}$, so rot_1 acts freely on $C_1(a)_t - \{(a,0,0)\} \supset C_1(a)_t^*$ with orbits of size $n_q(a) = |\omega|$.

If $a^2 = t+2$, then $C_1(a)_t$ is the degenerate conic $(y - \omega z)(y - \omega^{-1} z) = 0$, which consists of two lines intersecting at the origin. If $a^2 \neq t+2$, then $C_1(a)_t$ is a non-degenerate conic with two points at infinity, so $C_1(a)_t$ has $q-1$ points.

If $q = p \geq 3$ and $t = -2$, then the degenerate case $a^2 = t+2$ corresponds to $a = 0$, in which case $C_1(0)_{-2}^* = C_1(0)_{-2} - \{(0,0,0)\}$. In this case we obtain $\frac{2(p-1)}{4}$ orbits on $C_1(0)_{-2}^*$, each of size 4. This occurs if and only if $a^2 - 4 = -4 \in \mathbb{F}_p^\times$ is a square; equivalently, $p \equiv 1 \mod 4$. If $p \equiv 3 \mod 4$, then $a = 0$ is not hyperbolic. For non-degenerate hyperbolic a's, $|C_1(a)_{-2}^*| = p-1$ so $C_1(a)_{-2}^*$ contributes $\frac{p-1}{n_q(a)}$ orbits, each of size $n_q(a)$. Thus, if $q = p \geq 3$ and $t = -2$, the number of rot_1 orbits on $C_1(a)_{-2}^*$ for hyperbolic a's is

$$\left(\#\text{ of } \mathrm{rot}_1 \text{ orbits on } \coprod_{\substack{a \in \mathbb{F}_p \\ \text{hyperbolic}}} C_1(a)_{-2}^* \right)$$
$$= \begin{cases} \frac{2(p-1)}{4} + \sum_{\substack{d \mid p-1 \\ d \neq 1,2,4}} \frac{\phi(d)}{2} \cdot \frac{p-1}{d} & \text{if } p \equiv 1 \mod 4, \\ \sum_{\substack{d \mid p-1 \\ d \neq 1,2}} \frac{\phi(d)}{2} \cdot \frac{p-1}{d} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Here d should be thought of as $n_q(a) = |\omega|$.

(3) Suppose $a^2 - 4 \in \mathbb{F}_q^\times$ is a non-square, $t \neq 2$, and $q \geq 3$ is odd. In this case we say a is elliptic. Elliptic a's are in bijection with the set of polynomials of the form $T^2 - aT + 1$ which are non-split in $\mathbb{F}_q$ but have distinct roots in $\mathbb{F}_{q^2}^\times$. These roots are precisely the elements of $\mathbb{F}_{q^2}^\times$ which lie in the unique subgroup of order $q+1$ but not in the subgroup of order $q-1$, and hence there are $\frac{q-1}{2}$ elliptic a's.

In this case rot_1 acts via $\left[\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right]$ which is diagonalizable in $\mathbb{F}_{q^2}$ with distinct conjugate eigenvalues $\omega, \omega^{-1} \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q$, so $\omega^{-1} = \omega^q$, equivalently $\omega^{q+1} = 1$, and as usual $a = \omega + \omega^{-1}$. For the same reason as the hyperbolic case, we find rot_1 acts freely on $C_1(a)_t - \{(0,0,0)\} \supset C_1(a)_t^*$ with orbits of size $n_q(a) = |\omega|$.

If $a^2 = t+2$, then over $\mathbb{F}_{q^2}$, $C_1(a)_t$ is given by $(y - \omega z)(y - \omega^{-1} z) = 0$, but since $\omega \notin \mathbb{F}_q^\times$, $C_1(a)_t$ is empty. If $a^2 \neq t+2$, then $C_1(a)_t$ is a non-degenerate conic with $q+1$ points.

If $q = p \geq 3$ and $t = -2$, then the degenerate case $a^2 = t + 2$ (equivalently $a = 0$, equivalently $|\omega| = 4$) is elliptic if and only if $a^2 - 4 = -4 \in \mathbb{F}_p^\times$ is a non-square, equivalently $p \equiv 3 \mod 4$, in which case $C_1(0)^*_{-2}$ is empty. For every other elliptic a, $|C_1(a)^*_{-2}| = p + 1$ so we have $\frac{p+1}{n_q(a)}$ rot_1-orbits of size $n_q(a)$. Thus, if $q = p \geq 3$ and $t = -2$, the number of rot_1-orbits on $C_1(a)^*_{-2}$ for elliptic a's is

$$\left(\#\text{ of rot}_1\text{ orbits on } \bigsqcup_{\substack{a\in\mathbb{F}_p\\ \text{elliptic}}} C_1(a)^*_{-2}\right) = \begin{cases} \sum_{\substack{d|p+1\\ d\neq 1,2}} \frac{\phi(d)}{2}\cdot\frac{p+1}{d} & \text{if } p\equiv 1 \mod 4,\\ \sum_{\substack{d|p+1\\ d\neq 1,2,4}} \frac{\phi(d)}{2}\cdot\frac{p+1}{d} & \text{if } p\equiv 3 \mod 4.\end{cases}$$

Here d should be thought of as $n_q(a) = |\omega|$.

(4) Finally assume $q \geq 3$ is even and $t \neq 2$ (equivalently $t \neq 0$).
- If $a = 0 = -2 = 2$, then $C_1(0)_t$ is given by the equation $y^2 + z^2 = (y - z)^2 = t$. Here $\mathrm{rot}_1 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ fixes points on the line $y = z$ and acts with order 2 on every other vector in $\mathbb{F}_q^2$. Such fixed points lie in $C_1(0)_t$ only when $t = 0 = 2$, which we have excluded, so in this case $C_1(0)_t$ has q points and rot_1 acts freely on $C_1(0)_t$ with order $n_q(a) = 2 = p$.
- If $a \in \mathbb{F}_q^\times$, rot_1 acts via $\left[\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right]$ with characteristic polynomial $T^2 - aT + 1$. Since $a \neq 0$, this matrix has distinct eigenvalues $\omega, \omega + a \in \mathbb{F}_{q^2}^\times$ so again rot_1 is diagonalizable over $\mathbb{F}_q^\times$ and hence acts freely on $\mathbb{F}_q^2 - \{(0,0)\}$ and hence also acts freely on $C_1(a)_t - \{(a,0,0)\} \supset C_1(a)^*_t$ with orbits of size $n_q(a)$.

The above discussion implies that for any prime power $q \geq 3$, $a \in \mathbb{F}_q$, $t \neq 2 \in \mathbb{F}_q$, every orbit of rot_1 on $C_1(a)_t - \{(a,0,0)\} \subset \mathbb{F}_q^3$ has size $n_q(a)$ (Definition 5.1.4). In particular, this holds for every rot_1 orbit on $C_1(a)^*_t$ and $C_1(a)^\circ_t$. ☐

PROPOSITION 5.3.3. *For $q = p \geq 3$ a prime, $C_1(a)^*_{-2} = C_1(a)^\circ_{-2}$ is empty if and only if $a \in \{0, 2, -2\} \subset \mathbb{F}_p$ and $p \equiv 3 \mod 4$. Moreover, we have*

$$|\mathbb{X}^*(p)| = |X^\circ_{-2}(p)| = |X^*_{-2}(p)| = \begin{cases} p(p+3) & p \equiv 1 \mod 4,\\ p(p-3) & p \equiv 3 \mod 4.\end{cases}$$

Proof. From the proof of Lemma 5.3.1, we see that $C_1(a)^*_{-2}$ is always nonempty if $a \notin \{0, \pm 2\}$. If $a = 0$, then we are in the degenerate case $a^2 = t + 2$, in which case $C_1(a)^*_{-2}$ is empty if and only if $p \equiv 3 \mod 4$. If $a = \pm 2$, then a is parabolic and is again empty if and only if $p \equiv 3 \mod 4$.

To compute the cardinality of $X^*_{-2}(p)$, if $p \equiv 1 \mod 4$, then the cases $a = \pm 2$ contribute $4p$ solutions. The degenerate hyperbolic case $a = 0$ contributes $2(p-1)$ solutions, and the remaining $\frac{p-5}{2}$ non-degenerate hyperbolic a's contribute $(p-1)$ solutions each. Each of the $\frac{p-1}{2}$ elliptic a's is non-degenerate and contributes $p+1$ solutions each, so in total we get $p(p+3)$ solutions in this case. If $p \equiv 3 \mod 4$, then there are no parabolic a's, and each of the $\frac{p-3}{2}$ hyperbolic a's is non-degenerate. The case $a = 0$ is degenerate elliptic, giving zero solutions, leaving $\frac{p-3}{2}$ non-degenerate elliptic a's, so in this case we get $p(p-3)$ as desired. □

Definition 5.3.4. For a natural number n, let

$$\Phi(n) := \sum_{d|n} \frac{\phi(d)}{d}.$$

PROPOSITION 5.3.5. *For $q = p \geq 3$, the number of rot_1-orbits on $X^*_{-2}(p) = X^\circ_{-2}(p)$ is*

$$\left|X^*_{-2}(p)/\,\mathrm{rot}_1\right| = \begin{cases} \frac{p-1}{2}\Phi(p-1) + \frac{p+1}{2}\Phi(p+1) + \frac{-5p+11}{4} & \text{if } p \equiv 1 \mod 4, \\ \frac{p-1}{2}\Phi(p-1) + \frac{p+1}{2}\Phi(p+1) + \frac{-7p-1}{4} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Proof. This follows from the proof of Lemma 5.3.1. □

5.4. *Congruences for* $\mathrm{SL}_2(\mathbb{F}_q)$*-structures.* Let $q \geq 3$ be a prime power. Let $G \leq \mathrm{SL}_2(\mathbb{F}_q)$ be an absolutely irreducible subgroup which is not of dihedral type (Definition 5.2.14). Then we may speak of the trace of elements of G. Let $t \neq 2 \in \mathbb{F}_q$, and let $\mathcal{A}dm(G)_t \subset \mathcal{A}dm(G)$ be the open and closed substack classifying G-covers whose Higman invariants have trace t (Definition 5.2.19). Let $\mathcal{X} \subset \mathcal{A}dm(G)_t$ be a component, and let X be its coarse scheme. Here we will apply Theorem 3.5.1 to establish congruences on the degree of the map $X \to \overline{M(1)}$. We want to bound the integers $d_{\mathcal{X}}, m_{\mathcal{X}}$ which appear in Theorem 3.5.1.

First we bound $m_{\mathcal{X}}$. Let $\mathcal{C} \xrightarrow{\pi} \mathcal{E} \longrightarrow \mathcal{X}$ be the universal admissible $\mathrm{SL}_2(\mathbb{F}_q)$-cover over $\mathcal{X}$ with reduced ramification divisor $\mathcal{R}_\pi \subset \mathcal{C}$.

PROPOSITION 5.4.1. *Let $q \geq 3$ be a prime power. Let $t \in \mathbb{F}_q$ be a q-admissible trace. Then the automorphism groups of geometric points of $\mathcal{R}_\pi$ are all killed by* 24. *In the following cases we can do slightly better*:

(a) *If q is even, then the automorphism groups are killed by* 12.
(b) *If q is odd, $t \neq -2$, and $n_q(t)$ is odd, then the automorphism groups are killed by* 12.

In particular, in the language of Theorem 3.5.1, *we must have $m_{\mathcal{X}} \mid 24$ for q odd and $m_{\mathcal{X}} \mid 12$ in cases* (a) *or* (b) *above.*

Proof. This follows immediately from Theorem 5.3.2, noting that all automorphism groups in $\overline{\mathcal{M}(1)}$ are killed by 12. □

Next we bound $d_{\mathcal{X}}$. Let $\mathcal{R} \subset \mathcal{R}_\pi$ be a connected component. Then we wish to bound the degree of $\mathcal{R}/\mathcal{X}$. By the definition of the Higman invariant, the stabilizer in $\mathrm{SL}_2(\mathbb{F}_q)$ of a geometric point of $\mathcal{C}$ lying over the zero section of $\mathcal{E}$ is conjugate to $\langle c\rangle$. By Proposition 3.2.2, we find that $\mathcal{R}/\mathcal{X}$ is Galois with Galois group a subgroup of $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)/\langle c\rangle$. We can explicitly compute these centralizers as follows.

PROPOSITION 5.4.2. *Let $q = p^r \geq 3$ with p prime. Let $t \neq 2 \in \mathbb{F}_q$, and let $c \in \mathrm{SL}_2(\mathbb{F}_q) - \{\pm I\}$ be a non-central element with trace $\mathrm{tr}(c) = t$. The centralizers $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$ are as follows*:

(a) *If $t^2-4=0$, then $|c| = 2p$ and $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle) = \left\{\left[\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right] \mid a = \pm 1, b \in \mathbb{F}_q\right\} \cong \mathbb{F}_q \times \mu_2$. Thus $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$ has order $2q$.*

(b) *If t^2-4 is a square in $\mathbb{F}_q^\times$, then $|c| \mid q-1$ and $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$ is cyclic of order $q-1$.*

(c) *If t^2-4 is a non-square in $\mathbb{F}_q^\times$, then $|c| \mid q+1$ and $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$ is cyclic of order $q+1$.*

In cases (a) (*resp*. (b) *and* (c)) *we will say that c is parabolic* (*resp*. *hyperbolic*, *elliptic*). *Moreover* (a) *and* (c) *can only occur if q is odd. In particular*, *we find that if c is parabolic* (*resp*. *hyperbolic*, *elliptic*), *then in the language of Theorem 3.5.1*, *for any component $\mathcal{X} \subset \mathcal{A}dm(\mathrm{SL}_2(\mathbb{F}_q))_t$, $d_{\mathcal{X}}$ must divide p^{r-1}* (*resp*. $\frac{q-1}{|c|}$, $\frac{q+1}{|c|}$).

Proof. If $t^2-4=0$, then since $t \neq 2$, we must have $t = -2$, q odd, and $|c| = 2p$. By Proposition 5.1.3(c), c is conjugate to $\left[\begin{smallmatrix} -1 & u \\ 0 & -1\end{smallmatrix}\right]$ for some $u \in \mathbb{F}_q^\times$. An explicit calculation shows that its normalizer is the group of matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{F}_q)$ with $a^2 \in \mathbb{F}_p^\times$, and its centralizer is the subgroup with $a^2 = 1$.

For (b) and (c), we first make the following observation: If k is a field and $C \in \mathrm{GL}_d(k)$ is diagonalizable over k, then for any $A \in \mathrm{GL}_d(k)$, we have $ACA^{-1} = C^n$ if and only if A sends λ-eigenvectors to λ^n-eigenvectors. Indeed, if λ_v denotes the eigenvalue of an eigenvector $v \in k^d$, then

$$\begin{aligned} ACA^{-1} = C^n &\iff ACA^{-1}v = C^n v = \lambda_v^n v \quad \text{for all eigenvectors } v \in k^d \\ &\iff C(A^{-1}v) = \lambda_v^n(A^{-1}v) \quad \text{for all eigenvectors } v \in k^d. \end{aligned}$$

If t^2-4 is a square in $\mathbb{F}_q^\times$, then c is diagonalizable over $\mathbb{F}_q$ with distinct eigenvalues. If v, w is an eigenbasis, then A centralizes c if and only if $Av = \alpha v$, $Aw = \alpha^{-1}w$ for some $\alpha \in \mathbb{F}_q^\times$, so the centralizer is cyclic of order $q-1$.

If t^2-4 is a non-square in $\mathbb{F}_q^\times$, then c is not diagonalizable over $\mathbb{F}_q$ but is diagonalizable over $\mathbb{F}_{q^2}$ with distinct roots. By (b), we have $C_{\mathrm{SL}_2(\mathbb{F}_{q^2})}(\langle c\rangle)$ is cyclic of order q^2-1. Let σ denote the q-power Frobenius automorphism. Let $v \in \mathbb{F}_{q^2}^2$ be an eigenvector for c. Then we claim that v, v^σ is an eigenbasis for c.

If $v = (x, y) \neq (0,0)$, then $v^\sigma = av$ for $a \in \mathbb{F}_{q^2}^\times$ if and only if $x^q = ax, y^q = ay$, so $x^{q-1} = y^{q-1} = 1$, so $v \in \mathbb{F}_q^2$, which contradicts the assumption that c is not diagonalizable over $\mathbb{F}_q$, so $\{v, v^\sigma\}$ is an eigenbasis. If $A \in \mathrm{SL}_2(\mathbb{F}_q)$ centralizes c, then we must have

- $Av = \alpha v$ for some $\alpha \in \mathbb{F}_{q^2}^\times$, and
- $Av^\sigma = (Av)^\sigma = (\alpha v)^\sigma = \alpha^q v^\sigma$.

Since $\det(A) = 1$, we must have $\alpha^{q+1} = 1$. Conversely, the relation $(Av)^\sigma = Av^\sigma$ implies that any matrix in $\mathrm{GL}_2(\overline{\mathbb{F}_q})$ sending (v, v^σ) to $(\alpha v, \alpha^q v^\sigma)$ where $\alpha^{q+1} = 1$ must lie in $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$, so $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)$ is cyclic of order $q+1$. □

Plugging the above results into Theorem 3.5.1 gives us the following.

THEOREM 5.4.3. *Let $q = p^r \geq 3$ with p prime. Let $G \leq \mathrm{SL}_2(\mathbb{F}_q)$ be a subgroup which is not of dihedral type. Let $t \neq 2 \in \mathbb{F}_q$, and let $c \in \mathrm{SL}_2(\mathbb{F}_q) - \{\pm I\}$ be a non-central element with trace $\mathrm{tr}(c) = t$. Let $\mathcal{X} \subset \mathcal{A}dm(G)_t$ be a connected component with coarse scheme X. Then*

(a) *If $t^2 - 4 = 0$ and $q = p$, then $|c| = 2p$ and*

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod p.$$

(b) *If $t^2 - 4$ is a square in $\mathbb{F}_q^\times$, then*

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \frac{|c|}{\gcd\left(|c|, \frac{2(q-1)}{|c|}\right)}.$$

In particular, if $n \mid |c|$ is coprime to $\frac{2(q-1)}{|c|}$, then $\deg(X \to \overline{M(1)}) \equiv 0 \mod n$.

(c) *If $t^2 - 4$ is a non-square in $\mathbb{F}_q^\times$, then*

$$\deg(X \to \overline{M(1)}) \equiv 0 \mod \frac{|c|}{\gcd\left(|c|, \frac{2(q+1)}{|c|}\right)}.$$

In particular, if $n \mid |c|$ is coprime to $\frac{2(q+1)}{|c|}$, then $\deg(X \to \overline{M(1)}) \equiv 0 \mod n$.

If q is even, then $t^2 - 4$ must be a square in $\mathbb{F}_q^\times$, and then (b) *can be strengthened by replacing the $2(q-1)$ by $q-1$.*

Proof. Let $C \xrightarrow{\pi} \mathcal{E} \longrightarrow \mathcal{X}$ be the universal family, and let $\mathcal{R} \subset \mathcal{R}_\pi$ be a component of the reduced ramification divisor. By Proposition 3.2.2, the degree of $\mathcal{R}/\mathcal{X}$ divides the order of $C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)/\langle c\rangle$. Thus the theorem follows directly from Propositions 5.4.1 and 5.4.2 and Theorem 3.5.1. For the statement when q is even, note that $2 = 0 = -2 \in \mathbb{F}_q$ is never a q-admissible trace for q even, and every element of $\mathbb{F}_q^\times$ is a square. □

Using the "moduli interpretation" for $X_t^*(q)$ (see Theorem 5.2.10 and Definition 5.2.15), these congruences can be transported to congruences on the cardinality of $\mathrm{Aut}^+(\Pi)$-orbits on the sets $X_t^*(q)$.

THEOREM 5.4.4. *Let $q = p^r \geq 3$ with p prime. Let $t \neq 2 \in \mathbb{F}_q$. Let $n_q(t)$ denote the order of any non-central element of $\mathrm{SL}_2(\mathbb{F}_q)$ with trace t (cf. Definition 5.1.4 and Proposition 5.1.5). Let $P = (X, Y, Z)$ be an $\mathbb{F}_q$-point of the affine surface given by the equation*

$$x^2 + y^2 + z^2 - xyz = 2 + t.$$

Suppose that at least two of $\{X, Y, Z\}$ are non-zero. Let $\mathcal{O}$ be the $\mathrm{Aut}^+(\Pi)$-orbit of P.

(a) *If $t^2 - 4 = 0$ and $q = p \geq 3$, then*

$$|\mathcal{O}| \equiv 0 \mod p.$$

(b) *If $t^2 - 4$ is a square in $\mathbb{F}_q^\times$, then*

$$2|\mathcal{O}| \equiv 0 \mod \frac{n_q(t)}{\gcd\left(n_q(t), \frac{2(q-1)}{n_q(t)}\right)}.$$

(c) *If $t^2 - 4$ is a non-square in $\mathbb{F}_q^\times$, then*

$$2|\mathcal{O}| \equiv 0 \mod \frac{n_q(t)}{\gcd\left(n_q(t), \frac{2(q+1)}{n_q(t)}\right)}.$$

If q is even and $\mathcal{O} \subset X_t^\circ(q)$, then $t^2 - 4$ must be a square in $\mathbb{F}_q^\times$ and (b) can be strengthened by replacing $2|\mathcal{O}|$ by $|\mathcal{O}|$ and $2(q-1)$ by $q - 1$.

Proof. Since $t \neq 2$, by Proposition 5.2.16, our assumption on $\{X, Y, Z\}$ implies that $P \in X_t^*(q)$ (i.e., P is absolutely irreducible and not of dihedral type), so $\mathcal{O} \subset X_t^*(q)$. By Galois theory (Proposition 5.2.20), the orbit $\mathcal{O}$ corresponds to a connected component $\mathcal{M} \subset \mathcal{M}(G)_t/D(q, G)$ with degree over $\mathcal{M}(1)$ equal to $|\mathcal{O}|$, where G is the image of an absolutely irreducible representation $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$ not of dihedral type. Let M denote its coarse scheme. Then because γ_{-I} acts trivially on $X^*(q)$ (Proposition 5.2.20), $|\mathcal{O}|$ is also equal to the degree of $M \to M(1)$. We have maps

$$\mathcal{A}dm^0(G)_t \longrightarrow \mathcal{M}(G)_t \longrightarrow \mathcal{M}(G)_t/D(q, G),$$

where the first is rigidification by $Z(\mathrm{SL}_2(\mathbb{F}_q))$ (Proposition 2.5.10(b)), and the second is finite étale of degree $|D(q, G)|$. At the level of coarse schemes, the maps induce

$$Adm^0(G)_t \xrightarrow{\cong} M(G)_t \longrightarrow M(G)_t/D(q, G).$$

The first map is an isomorphism, and the second is finite flat with degree at most $|D(q, G)|$. By Propositions 5.2.13 and 6.4.1, we always have $|D(q, G)| \leq 2$.

Thus if X is any connected component of $Adm^0(G)_t$ mapping to M, then $\deg(X/M) \le 2$, so in this case we have

$$|\mathcal{O}| = \deg(\mathcal{M} \to \mathcal{M}(1)) = \deg(M \to M(1)) = \alpha \cdot \deg(X \to M(1)),$$

where $\alpha \in \{1, \frac{1}{2}\}$. If q is even and $G = \mathrm{SL}_2(\mathbb{F}_q)$ (equivalently $\mathcal{O} \subset X_t^\circ(q)$), then $D(q,G) = D(q)$ is trivial. Thus everything follows immediately from Theorem 5.4.3. ☐

Remark 5.4.5. When $G \le \mathrm{SL}_2(\mathbb{F}_q)$ is a proper subgroup, the bound $d_{\mathcal{X}} \mid |C_{\mathrm{SL}_2(\mathbb{F}_q)}(\langle c\rangle)/\langle c\rangle|$ can be improved to $d_{\mathcal{X}} \mid |C_G(\langle c\rangle)/\langle c\rangle|$. Using Proposition 6.4.1, this would give slight strengthenings of Theorems 5.4.3 and 5.4.4.

5.5. *Strong approximation for the Markoff equation.* The Markoff surface is the affine surface $\mathbb{M}$ defined by the Markoff equation

$$\mathbb{M} : x^2 + y^2 + z^2 - 3xyz = 0.$$

A Markoff triple is a positive integer solution to this equation, and a Markoff number is a positive integer that appears as a coordinate of a Markoff triple. Here we prove some results regarding $\mathbb{M}$. It will be convenient to use a twisted version of $\mathbb{M}$, which is isomorphic to $\mathbb{M}$ over $\mathbb{Z}[1/3]$ and has the benefit of admitting a moduli interpretation. Recall that taking trace coordinates induces an isomorphism $\mathrm{Tr} : X_{\mathrm{SL}_2} \xrightarrow{\sim} \mathbb{A}^3$, and $X_{\mathrm{SL}_2,-2} \subset X_{\mathrm{SL}_2}$ is the closed subscheme corresponding to representations with trace invariant -2 (Definition 5.2.15). Under the isomorphism Tr, $X_{\mathrm{SL}_2,-2}$ is isomorphic to the affine surface over $\mathbb{Z}$ given by

$$\mathbb{X} : x^2 + y^2 + z^2 - xyz = 0.$$

The map $\xi : (x,y,z) \mapsto (3x,3y,3z)$ defines a map

$$\xi : \mathbb{M} \longrightarrow \mathbb{X}$$

which induces an isomorphism of schemes $\xi : \mathbb{M}_{\mathbb{Z}[1/3]} \xrightarrow{\sim} \mathbb{X}_{\mathbb{Z}[1/3]}$. Since $\mathbb{X}(\mathbb{F}_3) = \{(0,0,0)\}$, ξ also induces a bijection on integral points

$$\xi : \mathbb{M}(\mathbb{Z}) \xrightarrow{\sim} \mathbb{X}(\mathbb{Z}).$$

Let $\mathbb{X}^*(p) := \mathbb{X}(\mathbb{F}_p) - \{(0,0,0)\}$, and let $\mathbb{M}^*(p) := \mathbb{M}(\mathbb{F}_p) - \{(0,0,0)\}$. Then by Proposition 5.2.5, the $\mathrm{Aut}(\Pi)$-action on $\mathbb{X}$ is generated by the automorphisms

$$\begin{array}{rcl} R_3 : (x,y,z) & \mapsto & (x,y,xy-z), \\ \tau_{12} : (x,y,z) & \mapsto & (y,x,z), \\ \tau_{23} : (x,y,z) & \mapsto & (x,z,y). \end{array}$$

Via ξ we obtain an induced action of $\mathrm{Aut}(\Pi)$ on $\mathbb{M}_{\mathbb{Z}[1/3]}$ generated by τ_{12}, τ_{23}, and $R_3' : (x,y,z) \mapsto (x,y,3xy-z)$, which extends to an action on $\mathbb{M}$ given by the same equations.

In [BGS16b], [BGS16a], Bourgain, Gamburd, and Sarnak study the action of $\mathrm{Aut}(\Pi)$ on $\mathbb{M}(\mathbb{F}_p)$ for primes p. They conjectured that for every prime p,

$\mathrm{Aut}(\Pi)$ acts transitively on $\mathbb{M}^*(p)$. One may check that for $p = 3$, $\mathrm{Aut}(\Pi)$ acts transitively on $\mathbb{M}^*(3)$ and $\mathbb{X}^*(3)$. (The latter is empty.) Thus, their conjecture is equivalent to

CONJECTURE 5.5.1 ([BGS16b], [BGS16a]). *For all primes p, $\mathrm{Aut}(\Pi)$ acts transitively on $\mathbb{X}^*(p)$.*

They were able to establish their conjecture for all but a sparse (though infinite) set of primes p:

THEOREM 5.5.2 ([BGS16a, Th. 2]). *Let $\mathbb{E}_{\mathrm{bgs}}$ denote the "exceptional set" of primes for which $\mathrm{Aut}(\Pi)$ fails to act transitively on $\mathbb{X}^*(p)$. For any $\epsilon > 0$,*

$$|\{p \in \mathbb{E}_{\mathrm{bgs}} \mid p \leq x\}| = O(x^\epsilon).$$

Moreover, they show that even if the conjecture were to fail, it cannot fail too horribly:

THEOREM 5.5.3 ([BGS16a, Th. 1]). *Fix $\epsilon > 0$. For every prime p, there is an $\mathrm{Aut}(\Pi)$-orbit $\mathcal{C}(p)$ such that*

$$|\mathbb{X}^*(p) - \mathcal{C}(p)| \leq p^\epsilon \qquad \text{for large } p$$

whereas $|\mathbb{X}^*(p)| = p(p+3)$ (resp. $p(p-3)$) if $p \equiv 1 \mod 4$ (resp. $3 \mod 4$).

Recall that we have defined $X^*_{-2}(q)$ as the subset of $\mathbb{X}(\mathbb{F}_q)$ which corresponds (via Theorem 5.2.10) to absolutely irreducible representations not of dihedral type (Definition 5.2.15). Thus Theorem 5.4.4(a) implies

THEOREM 5.5.4. *Every $\mathrm{Aut}^+(\Pi)$-orbit on $\mathbb{X}^*(p)$ has size divisible by p.*

Proof. By Proposition 5.2.17, for $p \geq 3$, $X^*_{-2}(p) = \mathbb{X}^*(p)$, so this case follows from Theorem 5.4.4(a). For $p = 2$, one can check the statement by hand. □

Combined with Theorem 5.5.3, this establishes Conjecture 5.5.1 for all but finitely many primes. Since the methods of [BGS16a] are effective, one obtains the following result:

THEOREM 5.5.5. *The exceptional set $\mathbb{E}_{\mathrm{bgs}}$ of Theorem 5.5.2 is finite and explicitly bounded.*

Recently, it was shown in [EFL+23] that every prime $p \in \mathbb{E}_{\mathrm{bgs}}$ satisfies $p < 3.448 \cdot 10^{392}$. (Conjecturally, $\mathbb{E}_{\mathrm{bgs}}$ is empty.)

Definition 5.5.6. Following [BGS16b], we say that an affine variety V over $\mathbb{Z}$ satisfies *strong approximation mod n* if the natural map $V(\mathbb{Z}) \to V(\mathbb{Z}/n\mathbb{Z})$ is surjective.

THEOREM 5.5.7. *In light of Theorem 5.5.5, we have*

(a) *for all but finitely many primes* p, $\mathrm{Aut}^+(\Pi)$ *acts transitively on* $\mathbb{X}^*(p)$ *and* $\mathbb{M}^*(p)$;
(b) *for all but finitely many primes* p, $M(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} := M(\mathrm{SL}_2(\mathbb{F}_p))_{-2}/D(p)$ *is connected*;
(c) *for all but finitely many primes* p, *both* $\mathbb{X}$ *and* $\mathbb{M}$ *satisfy strong approximation mod* p.

To be precise, (a), (b), *and* (c) *hold for every prime* $p \notin \mathbb{E}_{\mathrm{bgs}}$.

Proof. By the definition of $\mathbb{E}_{\mathrm{bgs}}$, for every $p \notin \mathbb{E}_{\mathrm{bgs}}$, $\mathrm{Aut}(\Pi)$ acts transitively on $\mathbb{X}^*(p)$. If $\mathrm{Aut}^+(\Pi)$ fails to act transitively, $\mathbb{X}^*(p)$ would consist of a pair of $\mathrm{Aut}^+(\Pi)$-orbits, and any automorphism of Π not lying in $\mathrm{Aut}^+(\Pi)$ would have to exchange the orbits, and hence act without fixed points. This is false, since for example the automorphism $(a,b) \mapsto (b,a)$ induces τ_{12} : $(x,y,z) \mapsto (y,x,z)$, which has the fixed point $(3,3,3) \in \mathbb{X}^*(p)$. Via ξ, this also establishes the transitivity on $\mathbb{M}^*(p)$. This proves (a). Part (b) is the Galois-theoretic translation of part (a) (see Proposition 5.2.20). Since the reduction map $\mathbb{X}(\mathbb{Z}) \to \mathbb{X}(\mathbb{F}_p)$ is $\mathrm{Aut}(\Pi)$-equivariant, (a) implies that the orbits of $(0,0,0)$ and $(3,3,3)$ map surjectively onto $\mathbb{X}(\mathbb{F}_p)$ for $p \notin \mathbb{E}_{\mathrm{bgs}}$. This result can be transported to $\mathbb{M}$ via ξ. This proves (c). □

Thus, we have effectively reduced Conjecture 5.5.1 to a finite computation. In [dCIL22], de-Courcy-Ireland and Lee have verified the conjecture for all primes $p < 3000$, so one expects the final computation to give a positive solution to the conjecture. If the computation indeed verifies the conjecture, then it follows from Proposition 5.3.3 that there are no congruence constraints on Markoff numbers mod p other than the ones first noted in [Fro13], namely that if $p \equiv 3 \mod 4$ and $p \neq 3$, then a Markoff number cannot be $\equiv 0, \frac{\pm 2}{3} \mod p$.

It follows from the work of Meiri-Puder [MP18] that the strong approximation property can be further extended to squarefree integers n whose prime divisors avoid the finite set $\mathbb{E}_{\mathrm{bgs}}$ and moreover satisfy the following condition:

$$\mathbf{MP}(p) := \begin{array}{l}\text{the property that either } p \equiv 1 \mod 4, \text{ or}\\ \text{the order of } \frac{3+\sqrt{5}}{2} \in \mathbb{F}_{p^2} \text{ is at least } 32\sqrt{p+1}.\end{array} \tag{5.8}$$

Thus $\mathbf{MP}(p)$ is satisfied for all primes $p \equiv 1 \mod 4$. Moreover by [MP18, Prop. A.1], we find that $\mathbf{MP}(p)$ holds for a density 1 set of primes. Their theorem is

THEOREM 5.5.8. *Let* $p_1, \ldots, p_r \notin \mathbb{E}_{\mathrm{bgs}}$ *be distinct primes such that for each* $i \in \{1, \ldots, r\}$, $p_i \notin \mathbb{E}_{\mathrm{bgs}}$ *and satisfies* $\boldsymbol{MP}(p_i)$. *Let* $n := p_1 p_2 \cdots p_r$. *The Chinese remainder theorem implies that* $\mathbb{X}(\mathbb{Z}/n\mathbb{Z}) = \mathbb{X}(\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times \mathbb{X}(\mathbb{Z}/p_r\mathbb{Z})$. *Let* $\mathbb{X}^*(n) \subset \mathbb{X}(\mathbb{Z}/n\mathbb{Z})$ *denote the solutions which do not reduce to the trivial solution* $(0,0,0) \mod p_i$ *for any* i. *Then we have*

(a) $\mathrm{Aut}(\Pi)$ *acts transitively on* $\mathbb{X}^*(n)$;
(b) *both* $\mathbb{X}$ *and* $\mathbb{M}$ *satisfies strong approximation mod* n.

Proof. Part (a) is [MP18, Cor. 1.7], which by induction on r implies that the $\mathrm{Aut}(\Pi)$-orbit space on $\mathbb{X}(\mathbb{Z}/n\mathbb{Z})$ is the product of the orbit spaces on $\mathbb{X}(\mathbb{Z}/p_i\mathbb{Z})$ for $i = 1, \dots, r$. This gives strong approximation for $\mathbb{X}$. The same argument gives strong approximation for $\mathbb{M}$ mod n, noting that if $3 \mid n$, then $\mathbb{M}(\mathbb{F}_3)$ has two singleton orbits $\{(0,0,0)\}$ and $\{(1,1,1)\}$. □

Finally, we note that Theorem 5.4.4 also gives a generalization of Theorem 5.5.4 to the $\mathbb{F}_q$-points of the Markoff-type equations $x^2+y^2+z^2-xyz = k$. For example, it implies

THEOREM 5.5.9. *Suppose* $\omega \in \mathbb{F}_{q^2}^\times$ *satisfies* $t := \omega + \omega^{-1} \in \mathbb{F}_q - \{2, -2\}$. *If* ℓ *is an odd prime and* $\mathrm{ord}_\ell(|\omega|) = r$ *and* $\mathrm{ord}_\ell(q(q^2-1)) = r + s$, *and* $P = (X, Y, Z)$ *is an* $\mathbb{F}_q$*-point of*

$$x^2 + y^2 + z^2 - xyz = t + 2$$

with at least two of $\{X, Y, Z\}$ *non-zero, then the* $\mathrm{Aut}^+(\Pi)$*-orbit of* P *has cardinality divisible by* $\ell^{\max\{r-s,0\}}$.

5.6. *A genus formula for* $M(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2,\overline{\mathbb{Q}}}$; *Finiteness of* $\mathrm{SL}_2(\mathbb{F}_p)$*-structures of trace invariant* -2. By Theorem 5.5.7 we find that the stack $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2,\overline{\mathbb{Q}}}$ is *connected* for primes p not lying in the finite set $\mathbb{E}_{\mathrm{bgs}}$. In this section we will establish a genus formula for (the compactification of) its coarse scheme M_p and show that for a density 1 set of primes, M_p is a "non-congruence modular curve." By Proposition 5.1.3(c′), the natural map $M(\mathrm{SL}_2(\mathbb{F}_p))_{-2} \to M_p$ is a totally split cover, and hence this will also compute the genus of the two isomorphic components of $M(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$.

For $p \geq 5$, let $\overline{M_p}$ denote the smooth compactification of the 1-dimensional scheme $M_p := M(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}$. (By Proposition 5.1.3, M_p is empty for $p = 2, 3$.) By Theorem 2.5.2(2), the forgetful map

$$\mathfrak{f} : \overline{M_p} \to \overline{M(1)} = \mathbb{P}^1_{\mathbb{Q}}$$

is finite étale over the complement of $j = 0, 1728, \infty$, where as usual we use the j-invariant to identify $\overline{M(1)}$ with the projective j-line. Thus, if $p \notin \mathbb{E}_{\mathrm{bgs}}$ (equivalently $\overline{M_p}$ is connected), then the genus of $\overline{M_p}$ can be computed by the Riemann-Hurwitz formula (see [Liu02, §7.4, Th. 4.16] or [Sil09, §II, Th. 5.9]). For this we need to know the degree of $\mathfrak{f}$ and the cardinalities of the ramified fibers $\mathfrak{f}^{-1}(0), \mathfrak{f}^{-1}(1728), \mathfrak{f}^{-1}(\infty)$. Using Proposition 5.2.20, all of these quantities can be calculated from the $\mathrm{Aut}^+(\Pi)$ action on $X^*_{-2}(p)$. For example, Proposition 5.3.3 implies that

$$\deg(\mathfrak{f}) = \begin{cases} p(p+3) & p \equiv 1 \mod 4, \\ p(p-3) & p \equiv 3 \mod 4. \end{cases}$$

The ramification above $j=0,1728$ was computed in [BBCL22, Prop. 3.3.2]:

PROPOSITION 5.6.1. *Let $p \geq 5$ be a prime.*

(a) *There is a unique unramified point in $\mathfrak{f}^{-1}(0)$. Every other point is ramified with index* 3, *so*

$$|\mathfrak{f}^{-1}(0)| = \begin{cases} \frac{p(p+3)-3}{3} & p \equiv 1 \mod 4, \\ \frac{p(p-3)-3}{3} & p \equiv 3 \mod 4. \end{cases}$$

(b) *There are precisely two unramified points in $\pi^{-1}(1728)$ if $p \equiv 1, 7 \mod 8$, and no unramified points otherwise. Every other point is ramified with index* 2. *Thus we obtain*

$$|\mathfrak{f}^{-1}(1728)| = \begin{cases} \frac{p^2+3p+2}{2} & \text{if } p \equiv 1 \mod 8, \\ \frac{p^2-3p}{2} & \text{if } p \equiv 3 \mod 8, \\ \frac{p^2+3p}{2} & \text{if } p \equiv 5 \mod 8, \\ \frac{p^2-3p+2}{2} & \text{if } p \equiv 7 \mod 8. \end{cases}$$

Recall that for an integer $n \geq 1$, we defined $\Phi(n) = \sum_{d|n} \frac{\phi(d)}{d}$. Theorem 2.5.2(4) (or Proposition 5.2.20) gives a combinatorial classification of the cusps. Using this, Proposition 5.3.5 calculates the number of cusps of $\overline{M}_p$:

PROPOSITION 5.6.2. *Let $p \geq 5$ be a prime. We have*

$$|\mathfrak{f}^{-1}(\infty)| = \begin{cases} \frac{p-1}{2}\Phi(p-1) + \frac{p+1}{2}\Phi(p+1) + \frac{-5p+11}{4} & \text{if } p \equiv 1 \mod 4, \\ \frac{p-1}{2}\Phi(p-1) + \frac{p+1}{2}\Phi(p+1) + \frac{-7p-1}{4} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

An elementary Riemann-Hurwitz calculation yields the following:

THEOREM 5.6.3. *Let $p \geq 5$ be prime. Let $\overline{M}_p$ be the smooth compactification of M_p. The smooth compactification of $M(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$ is a disjoint union of two copies of $\overline{M}_p$. For $p \notin \mathbb{E}_{\mathrm{bgs}}$, $\overline{M}_p$ is a smooth curve of genus*

$$\mathrm{genus}(\overline{M}_p) = \frac{1}{12}p^2 - \frac{p-1}{4}\Phi(p-1) - \frac{p+1}{4}\Phi(p+1) + \epsilon(p),$$

where

$$\epsilon(p) = \begin{cases} \frac{7}{8}p - \frac{29}{24} & \text{if } p \equiv 1 \mod 8, \\ \frac{5}{8}p + \frac{19}{24} & \text{if } p \equiv 3 \mod 8, \\ \frac{7}{8}p - \frac{17}{24} & \text{if } p \equiv 5 \mod 8, \\ \frac{5}{8}p + \frac{7}{24} & \text{if } p \equiv 7 \mod 8. \end{cases}$$

In particular, $\mathrm{genus}(\overline{M}_p) \sim \frac{1}{12}p^2$, *and moreover for all $p \geq 5$, we have*

$$\mathrm{genus}(\overline{M}_p) \geq \frac{1}{12}p^2 - \frac{1}{2}(p-1)^{3/2} - \frac{1}{2}(p+1)^{3/2} + \frac{1}{2}p.$$

Proof. Everything but the lower bound follows from the above discussion. The lower bound follows from the upper bound for the divisor function $d(n) \leq$

$2\sqrt{n}$, where $d(n)$ is the number of positive divisors of n, and noting that $\Phi(n) \leq d(n) \leq 2\sqrt{n}$, and $\epsilon(p) \geq \frac{1}{2}p$ for $p \geq 5$. $\square$

THEOREM 5.6.4. *For $p \geq 13$, $p \notin \mathbb{E}_{\mathrm{bgs}}$, $\overline{M}_p$ is a smooth curve of genus ≥ 2. For $p = 5, 7, 11$, $\overline{M}_p$ is a smooth curve of genus $0, 0, 1$ respectively. In particular, for $p \geq 13$, $p \notin \mathbb{E}_{\mathrm{bgs}}$ and any number field K, only finitely many elliptic curves over K admit a $\mathrm{SL}_2(\mathbb{F}_p)$-structure with ramification index $2p$ defined over K.*

Proof. By the classification of elements in $\mathrm{SL}_2(\mathbb{F}_p)$ as parabolic, hyperbolic, or elliptic, only parabolic elements have order $2p$, and by Proposition 5.1.3, this only happens when $p \geq 3$ and the trace is -2. The first part of the theorem follows from the lower bound for the genus obtained in Theorem 5.6.3, together with some explicit computer-aided calculations. The finiteness follows from the first part by Falting's theorem [CS86, §II]. $\square$

Given a connected finite étale cover $\mathfrak{f} : \mathcal{M} \to \mathcal{M}(1)_{\mathbb{C}}$ inducing $M \to M(1)_{\mathbb{C}}$ on coarse schemes, the analytic theory identifies $\mathcal{M}$ with a quotient $[\mathcal{H}/\Gamma]$, where $\mathcal{H}$ is the Poincaré upper half plane, and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup acting by Mobius transformations [Che18]. The subgroup Γ is uniquely determined by $\mathfrak{f}$ up to conjugation.

Definition 5.6.5. Let $k \subset \mathbb{C}$ be a subfield. We say that a finite étale map $\mathfrak{f} : \mathcal{M} \to \mathcal{M}(1)_k$ is a congruence modular stack (and $M \to M(1)_k$ is a congruence modular curve) if $\mathcal{M}_{\mathbb{C}}$ is connected and the subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ associated to $\mathfrak{f}_{\mathbb{C}}$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. (That is, Γ contains $\mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n))$ for some integer $n \geq 1$.) Otherwise we say that it is a non-congruence modular stack (resp. curve).

We end this section by showing that for a density 1 set of primes p, all components of $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))_{-2}$ (and hence $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))_{-2}$) are non-congruence.

Let Π, E, x_E be as in Situation 2.5.14. We work over $\overline{\mathbb{Q}}$. The action of $D(p) \cong \mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$ on $\mathrm{SL}_2(\mathbb{F}_p)$ descends via the characteristic quotient $h : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p)$ to an outer action on $\mathrm{PSL}_2(\mathbb{F}_p)$, where it also acts via the full outer automorphism group of $\mathrm{PSL}_2(\mathbb{F}_p)$. Thus h induces a finite étale surjection

$$\mathcal{M}(h) : \mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))/D(p) \longrightarrow \mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))/D(p)$$

and a surjection

$$h_* : \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))/D(p) \longrightarrow \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{PSL}_2(\mathbb{F}_p))/D(p)$$

which can be identified with the map on fibers above x_E induced by $\mathcal{M}(h)$. For $t \in \mathbb{F}_p$, let $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))_t$ denote the image of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))_t$. We say that $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))_t$ classifies $\mathrm{PSL}_2(\mathbb{F}_p)$-structures of trace invariant t. Since h is a central extension, any preimage of $x \in \mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))_t$ in $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))$ must also have trace invariant t, so there is no ambiguity. Let

$$\mathbb{P}_{\mathrm{MP}} := \{\text{primes } p \geq 5 \mid p \notin \mathbb{E}_{\mathrm{bgs}} \text{ and } p \text{ satisfies } \mathbf{MP}(p) \text{ (see (5.8))}\}.$$

Then $\mathbb{P}_{\mathrm{MP}}$ is a density 1 set of primes.

THEOREM 5.6.6 (Meiri-Puder). *We work over $\overline{\mathbb{Q}}$. For $p \in \mathbb{P}_{\mathrm{MP}}$, the stack $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} = \mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))_{-2}/D(p)$ is connected. Its degree over $\mathcal{M}(1)$ is*

$$d_p := \begin{cases} \frac{p(p+3)}{4} & p \equiv 1 \mod 4, \\ \frac{p(p-3)}{4} & p \equiv 3 \mod 4, \end{cases}$$

and its monodromy group over $\mathcal{M}(1)$ is the full alternating or symmetric group on d_p:

$$Mon(\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} \to \mathcal{M}(1)) = \begin{cases} S_{d_p} & \text{if } p \equiv 5,7,9,11 \mod 16, \\ A_{d_p} & \text{if } p \equiv 1,3,13,15 \mod 16. \end{cases}$$

Proof. Since $p \in \mathbb{P}_{\mathrm{MP}}$, $p \notin \mathbb{E}_{\mathrm{bgs}}$, the connectedness statement follows from the connectedness of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}$ (Theorem 5.5.7(b)). Let V denote the order 4 group of automorphisms acting freely on $\mathbb{X}^*(p)$ by negating two of the coordinates. Let $\mathbb{Y}^*(p) := \mathbb{X}^*(p)/V$. The action of Γ on $\mathbb{X}^*(p)$ descends to an action on $\mathbb{Y}^*(p)$, and the theory of the character variety allows us to identify the quotient map $\mathbb{X}^*(p) \to \mathbb{Y}^*(p)$ with the map

$$h_* : \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{-2}/D(p) \to \mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{PSL}_2(\mathbb{F}_p))_{-2}/D(p),$$

where the Γ-action is induced by the action of $\mathrm{Out}(\Pi)$ (Proposition 5.2.10). The sizes of $\mathbb{X}^*(p)$ were computed in Proposition 5.3.3. Thus the formula for d_p follows from the freeness of the V-action. By [MP18, Ths. 1.3, 1.4], the permutation image of Γ on $\mathbb{Y}^*(p)$ is either the full alternating or symmetric group. Since alternating groups in degrees ≥ 5 do not have non-trivial index 2 subgroups, the same is true for the permutation image of $\mathrm{Out}^+(\Pi)$ acting on $\mathrm{Epi}^{\mathrm{ext}}(\Pi, \mathrm{PSL}_2(\mathbb{F}_p))_{-2}/D(p)$. (Note $|\mathbb{Y}^*(p)| \geq 7$ for $p \geq 5$.) The determination of exactly when one obtains the alternating (or symmetric group) is done in [BBCL22, §3.3]. $\square$

Remark 5.6.7. Conjecturally, this theorem holds for all primes $p \geq 5$ [MP18, Conj. 1.2]

COROLLARY 5.6.8. *For $p \in \mathbb{P}_{\mathrm{MP}}$, the stack $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}$ is non-congruence. The same is true of $\mathcal{M}(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}$.*

Proof. First observe that a cover of a non-congruence modular stack is non-congruence, so it suffices to show that $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}$ is non-congruence.

If $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} \to \mathcal{M}(1)$ is congruence, then it fits into a factorization

$$\mathcal{M}(n) \to \mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} \to \mathcal{M}(1),$$

where $\mathcal{M}(n)$ is connected (in fact it is a component of $\mathcal{M}(\mathbb{Z}/n\times\mathbb{Z}/n)$) and the composition is Galois with Galois group $\mathrm{SL}_2(\mathbb{Z}/n)$. It follows that the monodromy group of $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}/\mathcal{M}(1)$ is a quotient of $\mathrm{SL}_2(\mathbb{Z}/n)$. Writing $n = \prod_{i=1}^r q_i$ with each q_i a prime power, we have $\mathrm{SL}_2(\mathbb{Z}/n) = \prod_{i=1}^r \mathrm{SL}_2(\mathbb{Z}/q_i)$, and hence the composition factors of $\mathrm{SL}_2(\mathbb{Z}/n)$ are either abelian or of the form $\mathrm{PSL}_2(\mathbb{F}_p)$ for primes $p \geq 5$, so the same must be true of the composition factors of the monodromy group of $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2}/\mathcal{M}(1)_{\mathbb{C}}$, but this contradicts Theorem 5.6.6. □

Remark 5.6.9. Theorem 5.6.3 can be viewed as a non-congruence analog of Rademacher's conjecture, proved by Dennin [DJ75], that there exist only finitely many congruence subgroups of a given genus. By contrast, it follows from Belyi's theorem that there are infinitely many non-congruence subgroups of every genus, so to obtain finiteness, one must restrict the types of non-congruence modular curves considered. Theorem 5.6.3 yields finiteness for the family $\{M(\mathrm{SL}_2(\mathbb{F}_p))^{\mathrm{abs}}_{-2} \mid p \in \mathbb{P}_{\mathrm{MP}}\}$.

Remark 5.6.10. In [Che18, Conj. 4.4.1], the author conjectured that for any non-solvable group G, the components of $\mathcal{M}(G)$ should all be non-congruence. While the above corollary and computational evidence strongly suggest this should be true for G an extension of $\mathrm{PSL}_2(\mathbb{F}_q)$ (for $q \geq 4$), the example of $\mathrm{PSU}_3(\mathbb{F}_4)$ from Section 4.11 implies that there exist four components of $\mathcal{M}(\mathrm{PSU}_3(\mathbb{F}_4))$ whose forgetful maps to $\mathcal{M}(1)$ are isomorphisms. In particular, they are congruence (with group $\mathrm{SL}_2(\mathbb{Z})$). Thus, this conjecture requires revision. A relatively safe revision is

CONJECTURE 5.6.11. *For any non-solvable group G, at least one component of $\mathcal{M}(G)$ is non-congruence. Equivalently, for any non-solvable group G, the kernel of the action of* $\mathrm{SL}_2(\mathbb{Z})\cong\mathrm{Out}^+(\Pi)$ *on* $\mathrm{Epi}^{\mathrm{ext}}(\Pi,G)$ *is non-congruence.*

This conjecture was proven in [Che18, Cor. 4.4.14] for any extension of S_n ($n \geq 4$), A_n ($n \geq 5$), $\mathrm{PSL}_2(\mathbb{F}_p)$ ($p \geq 5$), or any minimal non-abelian finite simple group.

In general it is an interesting and remarkably subtle problem to determine whether the components of $\mathcal{M}(G)$ are congruence or non-congruence. In geometric terms, this roughly asks

Which G-covers of elliptic curves can be expressed in terms of "torsion data"?

The example of $\mathrm{PSU}_3(\mathbb{F}_4)$ shows that there exist certain $\mathrm{PSU}_3(\mathbb{F}_4)$-covers of elliptic curves which are equivalent to the "trivial datum." One can also check computationally that there are components of $\mathcal{M}(\mathrm{PSU}_3(\mathbb{F}_5))$ which are congruence with group Γ, the unique index 3 normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with congruence level 3. In other words, certain $\mathrm{PSU}_3(\mathbb{F}_5)$-covers are, in an appropriate sense, equivalent to a configuration of 3-torsion points.

Despite these standouts, the general philosophy is that the "more non-abelian" G is, the "more non-congruence" $\mathcal{M}(G)$ should be. In [Che18, §4.2], we showed that for G a dihedral group, every component of $\mathcal{M}(G)$ is congruence. In a forthcoming work with Pierre Deligne, we show the same statement holds for metabelian groups G. The same statement is, however, false for groups of solvable length 3, and hence false for groups of solvable length n for any $n \geq 3$. Computationally, the components of $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_q))$ exhibit a striking numerical "regularity" which is not shared by many higher rank simple groups like $\mathrm{PSU}_3(\mathbb{F}_q)$. Given this, we tentatively conjecture that every component of $\mathcal{M}(\mathrm{PSL}_2(\mathbb{F}_q))$ is non-congruence. The reader is referred to the discussion in [Che18, §4] and the data in [Che18, App. B] for more details.

6. Appendix

6.1. *Normalized coordinates for tame balanced actions on prestable curves*. The purpose of this section is to prove Proposition 6.1.4 below, which shows that the étale local picture of an admissible G-cover (Definition 2.1.4) can be checked on fibers. The trickiest part is to check this at a node, where the calculation uses the explicit form of the projection maps onto isotypic subspaces (Lemma 6.1.2).

Remark 6.1.1 (Noetherian approximation). It is sometimes useful to prove results about prestable curves by working over a Noetherian base. By standard Noetherian approximation arguments, we do not lose any generality in doing so. A precise statement we will need is this. Let $C \to S$ be a prestable curve equipped with an effective Cartier divisor $R \subset C$ finite étale over a quasicompact quasiseparated scheme S and an S-linear action of a finite group G preserving R. Then we may write S as a limit $S = \lim_{i \in I} S_i$ with affine transition morphisms with each S_i of finite type over $\mathbb{Z}$ [Stacks, 01ZA,07RN]. In this case each map $S \to S_i$ is also affine [Stacks, 01YX]. Moreover, for some $i \in I$, the pair (C, R) with G-action is the base change of a prestable curve $C_i \to S_i$ with divisor $R_i \subset C_i$ finite étale over S_i and G-action preserving R_i. The key fact is that the category of schemes of finite presentation over S is the colimit of the categories of schemes of finite presentation over S_i [Stacks, 01ZM]. For example, from this, one can find a map $R_i \to C_i$ over S_i which pulls back to $R \to C$. By [Stacks, 0C5F,081C,081F], we may assume that C_i/S_i is a prestable curve, and that R_i/S_i is finite étale, and hence $R_i \to C_i$ must be the inclusion of an effective Cartier divisor. Similarly, viewing the G-action as being given as a collection of automorphisms satisfying certain properties, by [Stacks, 01ZM], we may assume the G-action is also the pullback of an S_i-linear G-action on (C_i, R_i).

Lemma 6.1.2. *Let A be a ring such that* $\operatorname{Spec} A$ *is connected. Let $e \geq 1$ be an integer invertible in A. Suppose the finite étale group scheme $\mu_{e,A} = \operatorname{Spec} A[x]/(x^e - 1)$ is totally split over A. Let M be an A-module, and let G be a cyclic group of order e generated by g, acting A-linearly on M. For a root of unity $\zeta \in \mu_e(A)$, let*

$$p_\zeta : M \longrightarrow M \qquad \textit{be given by} \qquad p_\zeta(m) = \frac{1}{e}\sum_{j=0}^{e-1} \zeta^{-j} g^j(m).$$

Then $p_\zeta(M) = M_\zeta := \{m \in M \mid gm = \zeta m\}$, and $\bigoplus_\zeta p_\zeta : M \to \bigoplus_{\zeta\in\mu_e(A)} M_\zeta$ is an isomorphism. Moreover, for each $\zeta \in \mu_e(A)$, formation of M_ζ defines an exact functor $\underline{\boldsymbol{Mod}}_{A[G]} \to \underline{\boldsymbol{Mod}}_{A[G]}$.

Proof. It is easy to check that p_ζ maps into M_ζ and that the composition $M_\zeta \hookrightarrow M \stackrel{p_\zeta}{\to} M_\zeta$ is the identity. Thus, each M_ζ is a direct summand of M. Moreover, if $\zeta_e \in A$ is a primitive e-th root of unity, then looking at the map $A[x]/(x^e - 1) \to A$ sending $x \mapsto \zeta_e$ shows that $\frac{1}{e}\sum_{\zeta\in\mu_e(A)} \zeta^{-j} = 1$ if $j \equiv 0 \mod e$, and is zero otherwise. Thus, the map

$$\begin{aligned} \oplus_\zeta p_\zeta : M &\longrightarrow \bigoplus_{\zeta\in\mu_e(A)} M_\zeta, \\ m &\mapsto \sum_{\zeta\in\mu_e(A)} p_\zeta(m) = \frac{1}{e}\sum_{\zeta\in\mu_e(A)}\sum_{j=0}^{e-1} \zeta^{-j} g^j(m) = \frac{1}{e}\sum_{j=0}^{e-1}\sum_{\zeta\in\mu_e(A)} \zeta^{-j} g^j(m) \end{aligned}$$

is the identity. This establishes the desired decomposition. The exactness of $M \mapsto M_\zeta$ is easy to check. ☐

Lemma 6.1.3 (Henselization commutes with G-invariants). *Let R be a ring equipped with an action of a finite group G (no tameness assumptions). Let $\mathfrak{m}_R \subset R$ be a maximal ideal. Let $\mathfrak{m}_{R^G} := \mathfrak{m}_R \cap R^G$. Let $(R^h, \mathfrak{m}_R^h), ((R^G)^h, \mathfrak{m}_{R^G}^h)$ denote the henselizations of the pairs $(R, \mathfrak{m}_R), (R^G, \mathfrak{m}_{R^G})$* [Stacks, 0A02]*. Then*

(a) *$\mathfrak{m}_{R^G} := \mathfrak{m}_R \cap R^G$ is a maximal ideal of R^G.*
(b) *$R^h, (R^G)^h$ are local rings with maximal ideals $\mathfrak{m}_R^h, \mathfrak{m}_{R^G}^h$.*
(c) *Let $f : (R^G)^h \to R^h$ be the natural map induced by the morphism of pairs $(R^G, \mathfrak{m}_{R^G}) \to (R, \mathfrak{m}_R)$. Then f induces an isomorphism $R \otimes_{R^G} (R^G)^h \xrightarrow{\sim} R^h$.*
(d) *The natural map $(R^G)^h = (R^G) \otimes_{R^G} (R^G)^h \longrightarrow (R \otimes_{R^G} (R^G)^h)^G \cong (R^h)^G$ is an isomorphism.*

Proof. Integral morphisms are universally closed, so we get (a). Henselization of a pair (A, I) preserves the quotient A/I [Stacks, 0AGU], so $R^h/\mathfrak{m}_R^h \cong R/\mathfrak{m}_R, (R^G)^h/\mathfrak{m}_{R^G}^h \cong R^G/\mathfrak{m}_{R^G}$ are fields, so $\mathfrak{m}_R^h, \mathfrak{m}_{R^G}^h$ are maximal. On the other hand, $\mathfrak{m}_R^h, \mathfrak{m}_{R^G}^h$ must be contained in the Jacobson radical of their corresponding rings [Stacks, 09XE], so $R^h, (R^G)^h$ are local, so we get (b). Since every $r \in R$ satisfies the polynomial $\prod_{g\in G}(T - gr) \in R^G[T]$, $R^G \to R$ is integral, so (c) is [Stacks, 0DYE]. Finally, henselization is flat, so (d) follows from the fact that taking G-invariants commutes with flat base change [KM85, Prop. A.7.1.3]. □

PROPOSITION 6.1.4 (Normalized coordinates for tame balanced actions on nodal curves). *Let G be a finite group with order invertible on S. Let C/S be a prestable curve equipped with a (S-linear) right action of G acting faithfully on fibers. Let k be an algebraically closed field and $\overline{p} : \operatorname{Spec} k \to C$ a geometric point with image $\overline{s}$ in S. Let $A := \mathcal{O}_{S,\overline{s}}$, and let $R := \mathcal{O}_{C_A,\overline{p}}$ the strict local ring of C_A at $\overline{p}$. Suppose the stabilizer $G_{\overline{p}}$ is cyclic of order e and acts faithfully on R. Let $g \in G_{\overline{p}}$ be a generator.*

(a) *Suppose the image of $\overline{p}$ is smooth inside $C_{\overline{s}}$. Then there is an A-algebra homomorphism*
$$\phi : A[z] \longrightarrow R$$
such that
- *ϕ induces an isomorphism between R and the strict local ring of $A[z]$ at $(\mathfrak{m}_A, z)$;*
- *there is a primitive e-th root of unity $\zeta_e \in A$ such that ϕ is $G_{\overline{p}}$-equivariant relative to the action of $G_{\overline{p}}$ on $A[z]$ given by $gz = \zeta_e z$.*

(b) *Suppose the image of $\overline{p}$ is a node in the fiber $C_{\overline{s}}$. The normalization of $C_{\overline{s}}$ defines a decomposition of the Zariski cotangent space $T^*_{C_{\overline{s}},\overline{p}}$ into a sum of two 1-dimensional subspaces, called "branches." Suppose the action of $G_{\overline{p}}$ on $T^*_{C_{\overline{s}},\overline{p}}$ preserves this decomposition and has image contained in $\mathrm{SL}(T^*_{C_{\overline{s}},\overline{p}})$. Then there are an element $a \in \mathfrak{m}_A$ and an A-algebra homomorphism*
$$\phi : A[z,w]/(zw - a) \longrightarrow R$$
such that
- *ϕ induces an isomorphism between R and the strict local ring of $A[z,w]/(zw-a)$ at $(\mathfrak{m}_A, z, w)$;*
- *there is a primitive e-th root of unity $\zeta_e \in A$ such that ϕ is $G_{\overline{p}}$-equivariant relative to the action of $G_{\overline{p}}$ on $A[z,w]/(zw-a)$ given by $gz = \zeta_e z$, $gw = \zeta_e^{-1} w$.*

In particular, in case (a) the quotient map $\pi : C \to C/G$ satisfies condition (4) of the definition of an admissible G-cover near $\overline{p}$ (Definition 2.1.4), and in case (b), the quotient map satisfies conditions (5) and (6) of the definition of an admissible G-cover near $\overline{p}$.

Proof. By Noetherian approximation (Remark 6.1.1), we may assume that S is of finite type over $\mathbb{Z}$.[34] Let κ be the residue field of A, and let $\overline{R} := R \otimes_A \kappa$.

[34] In particular every ring in the proof will be Noetherian. This makes it easier for us to work with completions.

We begin with part (a). By the local picture at a smooth point [Stacks, 054L], there is a map $A[x] \to R$ inducing an isomorphism between R and the strict local ring of $A[x]$ at the maximal ideal $(\mathfrak{m}_A, x)$. We abuse notation and also let x denote the image of x under the map $A[x] \to R$. Let $\widehat{R}$ denote the completion of R; then $\widehat{R} \cong \widehat{A}[\![x]\!]$. The Zariski cotangent space of $C_{\overline{s}}$ at $\overline{p}$ is the 1-dimensional κ-vector space $\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 = xR/(\mathfrak{m}_A, x^2)$. Since the G-action is faithful, g acts on the cotangent space by multiplication by a primitive e-th root of unity ζ_e. Let $\overline{x}$ denote the image of x in $\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$. By Lemma 6.1.2, the surjection $xR \to \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$ induces a surjection $(xR)_{\zeta_e} \to (\mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2)_{\zeta_e} = \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$. Let $x' \in (xR)_{\zeta_e}$ denote any preimage of $\overline{x}$. Then we have $x' = x + m + fx^2$ for some $m \in \mathfrak{m}_A R$ and $f \in R$. Because $\widehat{A}[\![x]\!]$ is also the completion of $A[x]$ at $(\mathfrak{m}_A, x)$, the universal property of power series rings gives us a unique $\widehat{A}$-algebra map $\widehat{A}[\![x]\!] \to \widehat{A}[\![x]\!]$ sending $x \mapsto x' = x + m + fx^2$. This map is, moreover, an automorphism since it is the composition of the automorphisms $x \mapsto x + m$ and $x \mapsto x + fx^2 = x(1 + fx)$. We will view this as giving an automorphism of $\widehat{R}$. Since R is a colimit of étale $A[x]$-algebras, there is an étale $A[x]$-algebra R_0 such that x' comes from an element of R_0. Then we have a commutative diagram

$$\begin{tikzcd} A[x] \arrow[r] \arrow[d, "x\mapsto z"'] & R_0 \arrow[r] & R \arrow[r, hook] & \widehat{R} \arrow[d, "x\mapsto x'"] \\ A[z] \arrow[r, "z\mapsto x'"] & R_0 \arrow[r] & R \arrow[r, hook] & \widehat{R}, \end{tikzcd}$$

where the unlabeled maps are the obvious ones. Since A is already strict henselian, $\widehat{R}$ is the completion of both R_0 and also of $A[x]$ at $(\mathfrak{m}_A, x)$. Thus we find that the map $A[z] \to R_0$ sending $z \mapsto x'$ induces an isomorphism on completions, and hence it is étale [Liu02, §4.3 Prop. 3.26], so the composition $\phi : A[z] \to R_0 \to R$ sending $z \mapsto x'$ identifies R with the strict local ring of $A[z]$ at $(\mathfrak{m}_A, z)$. By definition of x', ϕ is $G_{\overline{p}}$-equivariant relative to the action $gz = \zeta_e z$, as desired. Because A has separably closed residue field, by Lemma 6.1.3(d), the map $R^G \to R$ is G-equivariantly isomorphic to the map of henselizations induced by $A[z^e] = A[z]^G \hookrightarrow A[z]$, which shows that $\pi : C \to C/G$ satisfies condition (4) of the definition of an admissible G-cover.

Next we address (b). By the local picture at a node [Stacks, 0CBY], we find that R is the strict henselization of $A[x,y]/(xy-a)$ for some $a \in \mathfrak{m}_A$. Again we abuse notation and let x, y also denote their images in R. The cotangent space of the x-branch is $T_x^* := x\overline{R}/(x\overline{R} \cap \mathfrak{m}_{\overline{R}}^2)$. Since $G_{\overline{p}}$ acts faithfully, it acts on T_x^* by multiplication by a primitive e-th root of unity ζ_e. Since g preserves the branches of the node, it preserves the ideals xR, yR, so by Lemma 6.1.2, we obtain a surjection $(xR)_{\zeta_e} \to (T_x^*)_{\zeta_e} = T_x^*$. Let $x' \in (xR)_{\zeta_e}$ be any preimage of a basis of T_x^*. Then we must have $x' = ux$, where $u \notin (\mathfrak{m}_A R, xR, yR) = \mathfrak{m}_R$, so

$u \in R^\times$ is a unit. This implies that $u^{-1}y$ almost lies in $R_{\zeta_e^{-1}}$, in the sense that

$$uxu^{-1}y = xy = a = g(a) = g(xy) = g(uxu^{-1}y) = \zeta_e uxg(u^{-1}y). \tag{6.1}$$

Let $p_{\zeta_e^{-1}}$ be as in Lemma 6.1.2. Then by (6.1), $y' := p_{\zeta_e^{-1}}u^{-1}y$ satisfies

$$x'y' = \frac{x'}{e}\sum_{j=0}^{e-1}\zeta_e^j g^j(u^{-1}y) = \frac{1}{e}\sum_{j=0}^{e-1}\zeta_e^j uxg^j(u^{-1}y) = \frac{1}{e}\sum_{j=0}^{e-1}uxu^{-1}y = a.$$

On the other hand, y' and $u^{-1}y$ both map to the same basis element of $T_y^* := y\overline{R}/(y\overline{R}\cap\mathfrak{m}_{\overline{R}}^2)$, so $y' \equiv u^{-1}y \mod (\mathfrak{m}_A R, x^2R, y^2R)$. Arguing as in the smooth case, we find that there is an automorphism of $\widehat{R}$ sending $(x, y) \mapsto (x', y')$ (also see [Wew99, Prop. 2.1.1(ii)]), and such that the map $\phi : A[z, w]/(zw - a) \to R$ sending $(z, w) \mapsto (x', y')$ satisfies the desired properties. As in the smooth case, Lemma 6.1.3(d) implies that $R^G \hookrightarrow R$ is G-equivariantly isomorphic to the map on henselizations induced by $A[z^e, w^e]/(z^e w^e - a^e) \hookrightarrow A[z, w]/(zw - a)$, which shows that $\pi : C \to C/G$ satisfies conditions (5) and (6) of the definition of an admissible G-cover. □

6.2. *The isomorphism* $R[A, B, C] \xrightarrow{\sim} A[\Pi]^{\mathrm{GL}_{2,R}}$. Let Π be a free group of rank 2. For a ring R, let $A[\Pi] = A[\Pi]_R$ be the affine ring of $\mathrm{SL}_{2,R} \times \mathrm{SL}_{2,R}$. Thus $A[\Pi] = R[\{a_i, b_i, c_i, d_i\}_{i=1,2}]/(a_i d_i - b_i c_i - 1)_{i=1,2}$. Let $X_i := \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right]$, and let $A, B, C \in A[\Pi]$ be the functions $\mathrm{tr}(X_1), \mathrm{tr}(X_2), \mathrm{tr}(X_1X_2)$ respectively. Let $R[a, b, c]$ denote the polynomial ring on the variables a, b, c. The purpose of this section is to show that the map

$$\begin{aligned} f_R : R[a, b, c] &\longrightarrow A[\Pi]^{\mathrm{GL}_{2,R}}, \\ a, b, c &\mapsto A, B, C \end{aligned} \tag{6.2}$$

is an isomorphism for any ring R. The approach follows [BH95], though we fill in some missing details.

Lemma 6.2.1. *Let $T[\Pi] \subset A[\Pi]$ denote the R-subalgebra generated by the traces $\mathrm{tr}(X_{i_1}X_{i_2}\cdots X_{i_r})$ for all integers $r \geq 1$ with $i_j \in \{1, 2\}$. Then for any ring R, f_R is an isomorphism onto $T[\Pi]$.*

Proof. Surjectivity is a formal consequence of relations between traces, valid over any ring (see [BH95, Prop. 1.7]). Injectivity is [BH95, Prop. 3.2]. The trick here is to show that the images of A, B, C under a well-chosen R-algebra map are algebraically independent. Specifically, let $\mu_x := \left[\begin{smallmatrix} 1+x & x \\ 1 & 1\end{smallmatrix}\right]$. Let $\varphi : A[\Pi] \to R[x, y, z]$ be the R-algebra homomorphism given by $X_1 \mapsto \mu_x$ and $X_2 \mapsto \mu_z\mu_y\mu_z^{-1}$. We check that

$$\begin{aligned} &\varphi(A) = \mathrm{tr}(\mu_x) = 2 + x, \quad \varphi(B) = \mathrm{tr}(\mu_y) = 2 + y, \text{ and} \\ &\varphi(C) = \mathrm{tr}(\mu_x\mu_z\mu_y\mu_z^{-1}) = 2 + (x + y)(2 + z) - z^2 \end{aligned}$$

but these are clearly independent in $R[x,y,z]$, so A, B, C must be algebraically independent. $\square$

It remains to show that $T[\Pi] = A[\Pi]^{\mathrm{GL}_{2,R}}$. For an integer $n \geq 1$, let $A[n] = A[n]_R$ denote the R-algebra representing the functor $M_2^n : \underline{\mathbf{Alg}}_R \longrightarrow \underline{\mathbf{Sets}}$ sending an R-algebra S to the set $M_2(S)^n$. Thus $A[n]$ is a polynomial ring in the $4n$ variables $\{A_i, B_i, C_i, D_i \mid i \in \{1, \ldots, n\}\}$. Denote by Y_i the matrix variable $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$. Let $\mathrm{GL}_{2,R}$ act on $A[n]$ by conjugation on M_2^n.

Theorem 6.2.2 (First fundamental theorem of invariant theory for matrices). *For any ring R, the invariant ring $A[n]^{\mathrm{GL}_{2,R}}$ is generated as a R-algebra by $\mathrm{tr}(Y_{i_1} Y_{i_2} \cdots Y_{i_r})$ and $\det(Y_i)$, where i, i_j vary over $\{1, \ldots, n\}$ and r varies over positive integers.*

Proof. When $R = \mathbb{Z}$ or R is an algebraically closed field, this is due to Donkin (see [Don90, §3] or [DCP17, §15.2]). This can be bootstrapped to the general case as follows: the universal coefficient theorem [Jan03, I, Prop. 4.18] gives an exact sequence

(6.3)
$$0 \longrightarrow A[n]_{\mathbb{Z}}^{\mathrm{GL}_{2,\mathbb{Z}}} \otimes_{\mathbb{Z}} R \longrightarrow A[n]^{\mathrm{GL}_{2,R}} \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H^1(\mathrm{GL}_{2,\mathbb{Z}}, A[n]_{\mathbb{Z}}), R) \longrightarrow 0.$$

Thus it would suffice to show that $H^1(\mathrm{GL}_{2,\mathbb{Z}}, A[n]_{\mathbb{Z}})$ is $\mathbb{Z}$-flat, or equivalently that $\mathrm{Tor}_1^{\mathbb{Z}}(H^1(\mathrm{GL}_{2,\mathbb{Z}}, A[n]_{\mathbb{Z}}), \mathbb{F}_p) = 0$ for all primes p [Har10, §1, Lemma 2.1]. Since $\overline{\mathbb{F}_p}$ is faithfully flat over $\mathbb{F}_p$, it suffices to check the vanishing of the Tor group with $\mathbb{F}_p$ replaced by $\overline{\mathbb{F}_p}$, but this follows from (6.3) combined with Donkin's result over algebraically closed fields.[35] $\square$

The following lemma implies that f_R is an isomorphism, as desired.

Lemma 6.2.3. *For any ring R, the natural map $A[2] \to A[\Pi]$ sending $Y_i \mapsto X_i$ induces a surjection $\rho_2 : A[2]^{\mathrm{GL}_{2,R}} \to A[\Pi]^{\mathrm{GL}_{2,R}}$. In particular, $A[\Pi]^{\mathrm{GL}_{2,R}} = T[\Pi]$.*

Proof. Note that if the map $A[2] \to A[\Pi]$ admits a GL_2-equivariant section, then we are done. This is not quite true, but at least if $\Delta_i := \det(Y_i)$ and $\delta_i := \sqrt{\Delta_i}$, then there exists an equivariant map $q : A[\Pi] \to A[2][\delta_1^{\pm 1}, \delta_2^{\pm 1}]$ sending $a_i, b_i, c_i, d_i \mapsto A_i/\delta_i, B_i/\delta_i, C_i/\delta_i, D_i/\delta_i$. This turns out to be enough; see [BH95, §9, pp. 97–98, "surjectivity of ρ_n"]. $\square$

[35] In fact, we have $H^1(\mathrm{GL}_{2,\mathbb{Z}}, A[n]_{\mathbb{Z}}) = 0$. This is because the $\mathrm{GL}_{2,\mathbb{Z}}$-module $A[n]_{\mathbb{Z}}$ has a *good filtration* [Jan03, II, §4.16]. Donkin shows the existence of a good filtration over algebraically closed fields, but this implies that $A[n]_{\mathbb{Z}}$ also has a good filtration, which implies the vanishing of all higher cohomology [Jan03, II, Lemma B.9]. (To use Lemma B.9 one should filter $A[n]_{\mathbb{Z}}$ by $\mathbb{Z}$-finite $\mathrm{GL}_{2,\mathbb{Z}}$-submodules. See [Jan03, II, §8] for the definition and some properties of the Weyl modules $V(\lambda)$ over $\mathbb{Z}$.)

6.3. *The normalizer of* $\mathrm{SL}_n(\mathbb{F}_q)$ *in* $\mathrm{GL}_n(\overline{\mathbb{F}_q})$.

PROPOSITION 6.3.1. *Let q be a prime power, and let $n \geq 1$ be an integer. Then*

$$N_{\mathrm{GL}_n(\overline{\mathbb{F}_q})}(\mathrm{SL}_n(\mathbb{F}_q)) = \overline{\mathbb{F}_q}^\times \cdot \mathrm{GL}_n(\mathbb{F}_q) = \{uA \mid u \in \overline{\mathbb{F}_q}^\times, A \in \mathrm{GL}_n(\mathbb{F}_q)\}.$$

Moreover, the same is true if $\mathrm{SL}_n(\mathbb{F}_q)$ *is replaced by* $\mathrm{GL}_n(\mathbb{F}_q)$.

Proof. Certainly every matrix of the form uA with $u \in \overline{\mathbb{F}_q}^\times, A \in \mathrm{GL}_n(\mathbb{F}_q)$ normalizes $\mathrm{SL}_n(\mathbb{F}_q)$, so it remains to show that any matrix which normalizes must take this form. Let $\phi \in \mathrm{Aut}(\mathrm{GL}_n(\overline{\mathbb{F}_q}))$ denote the Frobenius automorphism defined on matrices by acting on coefficients via $a \mapsto a^q$. Then, for $A \in \mathrm{GL}_n(\overline{\mathbb{F}_q})$, $A \in \mathrm{GL}_n(\mathbb{F}_q)$ if and only if $\phi(A) = A$. Thus, for $B \in \mathrm{SL}_n(\mathbb{F}_q)$, we have $ABA^{-1} \in \mathrm{SL}_n(\mathbb{F}_q)$ if and only if $\phi(ABA^{-1}) = \phi(A)B\phi(A)^{-1} = ABA^{-1}$, which happens if and only if $A^{-1}\phi(A)$ centralizes B. Thus, A normalizes $\mathrm{SL}_n(\mathbb{F}_q)$ if and only if $A^{-1}\phi(A)$ centralizes $\mathrm{SL}_n(\mathbb{F}_q)$, which by Schur's lemma happens if and only if $A^{-1}\phi(A) \in \overline{\mathbb{F}_q}$ is a scalar. Equivalently, this is to say that $\phi(A) = uA$ for some unit $u \in \overline{\mathbb{F}_q}^\times$. It remains to characterize the elements of $\mathrm{SL}_n(\overline{\mathbb{F}_q})$ on which ϕ acts by multiplication by a unit $u \in \overline{\mathbb{F}_q}^\times$.

Suppose $A = (a_{ij}) \in \mathrm{GL}_n(\overline{\mathbb{F}_q})$ satisfies $\phi(A) = uA$ with $u \in \overline{\mathbb{F}_q}^\times$. Let $r \in \{a_{ij}\}$ be chosen so that $r \neq 0$. Then since $a_{ij}^q = ua_{ij}$, it follows that $\phi\left(\frac{1}{r}A\right) = \frac{1}{r}A$, so $\frac{1}{r}A \in \mathrm{GL}_n(\mathbb{F}_q)$, so we may write

$$A = rA', \qquad \text{where } r \in \overline{\mathbb{F}_q}^\times, A' \in \mathrm{GL}_n(\mathbb{F}_q)$$

as desired. □

6.4. *Images of absolutely irreducible representations* $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$. Let Π be a free group of rank 2. In [Mac69], Macbeath classified the possible images of absolutely irreducible representations $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$. Here, $\mathrm{PSL}_2(\mathbb{F}_q)$ denotes the quotient of $\mathrm{SL}_2(\mathbb{F}_q)$ by the subgroup of scalar matrices, and $\mathrm{PGL}_2(\mathbb{F}_q)$ denotes the quotient of $\mathrm{GL}_2(\mathbb{F}_q)$ by the subgroup of scalar matrices.

PROPOSITION 6.4.1. *Let $q = p^r$ be a prime power. Let $\varphi : \Pi \to \mathrm{SL}_2(\mathbb{F}_q)$ be an absolutely irreducible representation. Let $G := \varphi(\Pi)$, and let $\overline{G}$ be its image in $\mathrm{PSL}_2(\mathbb{F}_q)$. Let $Z := Z(\mathrm{SL}_2(\mathbb{F}_q))$ be the center, so Z has order 2 for q odd and is trivial for q even. Then $G, \overline{G}$ must fall into one of the following categories*:

(E1) $\overline{G} \cong D_{2n}$ *is dihedral* $(n \geq 2)$.
(E2) $\overline{G} \cong A_4$.
(E3) $\overline{G} \cong A_5$.
(E4) $\overline{G} \cong S_5$.
(P1) $\overline{G}$ *is* $\mathrm{PGL}_2(\mathbb{F}_q)$*-conjugate to* $\mathrm{PSL}_2(\mathbb{F}_{q'})$ *for some* $q' \mid q$. *In this case G is* $\mathrm{GL}_2(\mathbb{F}_q)$*-conjugate to* $\mathrm{SL}_2(\mathbb{F}_{q'})$.

(P2) *$\overline{G}$ is $\mathrm{PGL}_2(\mathbb{F}_q)$-conjugate to $\mathrm{PGL}_2(\mathbb{F}_{q'})$ for some q' satisfying $q'^2 \mid q$. In this case if q is odd, then G is $\mathrm{GL}_2(\mathbb{F}_q)$-conjugate to $\langle \mathrm{SL}_2(\mathbb{F}_{q'}), \left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right]\rangle$, where $a \in \mathbb{F}_{q'^2} - \mathbb{F}_{q'}$ with $a^2 \in \mathbb{F}_{q'}$. If q is even, then $\mathrm{PGL}_2(\mathbb{F}_{q'}) = \mathrm{PSL}_2(\mathbb{F}_{q'})$ and G must be as in case* (P1).

Moreover, we have the following:

(a) *In each of the above cases, G is a central extension of $\overline{G}$ by Z.*
(b) *In each of the above cases except* (E1) *(dihedral), let $N := N_{\mathrm{GL}_2(\mathbb{F}_q)}(G)$ be its normalizer and $C := C_{\mathrm{GL}_2(\mathbb{F}_q)}(G)$ be its centralizer. Then N/C has order at most* 2.

Proof. The cases (E1)–(E4) are called *exceptional*, and the cases (P1) and (P2) are called *projective*. They are not necessarily mutually exclusive. We begin with (a). Since Z is trivial for q even, we may assume q odd. In this case, an element $g \in \mathrm{PSL}_2(\mathbb{F}_q)$ of even order must either be diagonalizable over $\mathbb{F}_{q^2}$ or be conjugate to $\left[\begin{smallmatrix} -1 & u \\ 0 & -1\end{smallmatrix}\right]$ for $u \in \mathbb{F}_q^\times$. If g is diagonalizable over $\mathbb{F}_{q^2}$ of order $2k$, then $g^k = -I$, so $Z \subset G$. If g is conjugate to $\left[\begin{smallmatrix} -1 & u \\ 0 & -1\end{smallmatrix}\right]$, then $g^p = -I$, so $Z \subset G$.

For (b), we will proceed case by case. First suppose q is even, so $Z = 1$ and $G \cong \overline{G}$. Then N/C is naturally a subgroup of $\mathrm{Out}(G)$, but $\mathrm{Out}(A_4) \cong \mathrm{Out}(A_5)$ have order 2, and $\mathrm{Out}(S_5)$ is trivial. Finally, from Proposition 6.3.1, we know that $\mathrm{GL}_2(\mathbb{F}_q)$ acts on $\mathrm{SL}_2(\mathbb{F}_{q'})$ via $\mathrm{GL}_2(\mathbb{F}_{q'})$, so in this case we also have $|N/C| \le 2$. Now suppose q is odd, so G is a central extension of $\overline{G}$ by $Z \cong \mathbb{Z}/2\mathbb{Z}$. Case (P1) proceeds exactly as in the case where q is even. In case (P2), again we may assume $\overline{G} = \mathrm{PGL}_2(\mathbb{F}_{q'})$ and $G = \langle \mathrm{SL}_2(\mathbb{F}_{q'}), \left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right]\rangle$. We know that the image of N in $\mathrm{PGL}_2(\mathbb{F}_q)$ must normalize $\mathrm{PGL}_2(\mathbb{F}_{q'})$, hence it normalizes its unique index 2 subgroup $\mathrm{PSL}_2(\mathbb{F}_{q'})$, so N must normalize $\mathrm{SL}_2(\mathbb{F}_{q'})$, but then Proposition 6.3.1 implies that N acts via $\mathrm{GL}_2(\mathbb{F}_{q'})$, so again we must have $|N/C| \le 2$.

In cases (E2)–(E4), for each of $\overline{G} = A_4, A_5, S_5$, we will compute the outer automorphism groups of central extensions of $\overline{G}$ by Z. Using the exact sequence [Wei94, Exercise 6.1.5]

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_1(G,\mathbb{Z}), A) \to H^2(G, A) \to \mathrm{Hom}(H_2(G,\mathbb{Z}), A) \to 0,$$

we find that $H^2(A_4, Z), H^2(A_5, Z), H^2(S_5, Z)$ have orders 2, 2, and 4 respectively. Using GAP we explicitly construct the associated central extensions, and we find that in all cases except for the split extension $S_5 \times Z$, the extension has an outer automorphism group of order 2. However, we claim that if $\overline{G} \cong S_5$, then G cannot be isomorphic to $S_5 \times Z$. Indeed, in this case there is an element $\overline{g} \in \overline{G}$ of order 4, but for q odd, $\overline{g}$ must be the image of a matrix $g \in \mathrm{SL}_2(\mathbb{F}_q)$ which is diagonalizable over $\overline{\mathbb{F}_q}$, so g must have order 8, but $S_5 \times Z$ does not contain any elements of order 8. This shows that if $\overline{G} = A_4, A_5, S_5$, then $\mathrm{Out}(G) \cong \mathbb{Z}/2\mathbb{Z}$, so $|N/C| \le 2$.

Finally, we show that $G, \overline{G}$ must fall into one of the categories listed. Note that since φ is absolutely irreducible, by Lemma 5.2.9, we must have $\operatorname{tr}\varphi([a,b]) \neq 2$, so φ must be "non-singular" in the terminology of the first proof of Theorem 5.2.10, and hence [Mac69, Th. 2] implies that $\overline{G}$ cannot be an "affine group" in the sense of [Mac69, §4]. Thus the classification in [Mac69, §4] implies that $\overline{G}$ must fall into one of the categories described above. We can rule out the case $\overline{G} \cong D_2$ because in that case G would be abelian. It remains to show that G has the stated form in cases (P1) and (P2), but this follows from comparing cardinalities. □

6.5. *Étale local rings of Deligne-Mumford stacks*. The goal of this section is to show that (the completion of) the étale local ring of a geometric point of a Deligne-Mumford stack is canonically isomorphic to the universal deformation ring at that point. While this is certainly well known to experts, the author does not know of a good reference. The discussion here parallels the development in the stacks project [Stacks, 06G7], but we do *not* make the assumption that k is a *finite* Λ-algebra (see below). We do this so we can easily talk about the universal deformation rings of geometric points of $\mathcal{M}$.

6.5.1. *Étale local rings*. We begin with a discussion of étale local rings.

Definition 6.5.1. Let $\mathcal{M}$ be a Deligne-Mumford stack whose diagonal is representable (by schemes).[36] Let Ω be a separably closed field, and let $x : \operatorname{Spec}\Omega \to \mathcal{M}$ be a point. An étale neighborhood of x is a quadruple $(U, i, \tilde{x}, \alpha)$, where U is an affine scheme, $U, i, \tilde{x}$ form a diagram

$$\begin{tikzcd} \operatorname{Spec}\Omega \arrow[r, "\tilde{x}"] \arrow[dr, "x"'] & U \arrow[d, "i"] \\ & \mathcal{M}, \end{tikzcd}$$

and α is an isomorphism $x \xrightarrow{\sim} i \circ \tilde{x}$ in $\mathcal{M}(\operatorname{Spec}\Omega)$. A morphism of neighborhoods $(U, i, \tilde{x}, \alpha) \to (U', i', \tilde{x}', \alpha')$ is a pair (f, β), where f is a map $f : U \to U'$ and β is an isomorphism $\beta : i \xrightarrow{\sim} i' \circ f$ in $\mathcal{M}(U)$ such that all 2-morphisms in the associated "tetrahedron" are compatible. Note that given f, if there exists a β making (f, β) into a morphism of neighborhoods, then β is unique. Let $N_{\mathcal{M},x}$ denote the category of étale neighborhoods of x. Because $\mathcal{M}$ has representable diagonal, the same argument as in the schemes case shows that $N_{\mathcal{M},x}$ is cofiltered [Stacks, 03PQ]. The étale local ring of $\mathcal{M}$ at x is

$$\mathcal{O}_{\mathcal{M},x} := \operatorname{colim}_{(U,i,\tilde{x},\alpha)} \Gamma(U, \mathcal{O}_U),$$

[36] This implies that if $f : U \to \mathcal{M}$, $g : V \to \mathcal{M}$ are any maps from schemes, then $U \times_{\mathcal{M}} V$ is a scheme; i.e., f, g is are representable (by schemes).

where the colimit runs over the category of étale neighborhoods $N_{\mathcal{M},x}$. If $\mathcal{M}$ is a scheme, then $\mathcal{O}_{\mathcal{M},x}$ is the strict henselization of the local ring of the image of x [Stacks, 04HX]. The inclusion functor $\Phi : \underline{\mathbf{AffSch}} \hookrightarrow \underline{\mathbf{Sch}}$ is right adjoint to the functor $X \mapsto \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ [Stacks, 01I1], so Φ preserves all limits. Thus $\operatorname{Spec}\mathcal{O}_{\mathcal{M},x}$ is also the limit of all étale neighborhoods of x. Let $\kappa(x)$ denote the residue field of $\mathcal{O}_{\mathcal{M},x}$. For any étale neighborhood $(U, i, \tilde{x}, \alpha)$, $\mathcal{O}_{\mathcal{M},x}$ fits into a canonical 2-commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\Omega \arrow[r] \arrow[rr, bend left, "\lim \tilde{x}"] \arrow[rrd, bend right, "x"'] & \operatorname{Spec}\kappa(x) \arrow[r] & \operatorname{Spec}\mathcal{O}_{\mathcal{M},x} \arrow[r] \arrow[d, "i_x"] & U \arrow[ld, "i"] \\
& & \mathcal{M}. &
\end{tikzcd}$$

PROPOSITION 6.5.2. *Let $\mathcal{M}$ be a Deligne-Mumford stack with representable diagonal. Let Ω be a separably closed field, and let $x : \operatorname{Spec}\Omega \to \mathcal{M}$ be a point. The canonical map $i_x : \operatorname{Spec}\mathcal{O}_{\mathcal{M},x} \to \mathcal{M}$ is formally étale. If $(U, q, \tilde{x}, \alpha)$ is an étale neighborhood of x and u is the image of $\tilde{x}$, then there is a canonical isomorphism $\mathcal{O}_{\mathcal{M},x} \cong \mathcal{O}_{U,\tilde{x}}$ which induces an isomorphism between $\kappa(x)$ and the separable closure of $\kappa(u)$ inside Ω.*

Proof. First note that since $\mathcal{M}$ is Deligne-Mumford, étale neighborhoods *exist*. Thus the map i_x is a limit of étale morphisms, so it is formally étale. The strict henselization $\mathcal{O}_{U,\tilde{x}}$ is defined as the (global sections) of the cofiltered limit of étale neighborhoods of $\tilde{x} : \operatorname{Spec}\Omega \to U$. The category of étale neighborhoods of $\tilde{x}$ embeds into the category of étale neighborhoods of x, and since every étale neighborhood of x is refined by an étale neighborhood of $\tilde{x}$, this map induces an isomorphism on limits, whence the isomorphism $\mathcal{O}_{\mathcal{M},x} \cong \mathcal{O}_{U,\tilde{x}}$. The final statement follows from the fact that the residue field of $\mathcal{O}_{U,\tilde{x}}$ is the separable closure of $\kappa(u)$ inside Ω. □

6.5.2. *Étale local rings vs universal deformation rings*. We work universally over a scheme $\mathbb{S}$. Let $s : \operatorname{Spec} k \to \mathbb{S}$ be a morphism with k a field. Suppose it factors as $\operatorname{Spec} k \to \operatorname{Spec}\Lambda \subset \mathbb{S}$, where $\operatorname{Spec}\Lambda \subset \mathbb{S}$ is a Noetherian open affine subscheme. Let $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda,k}$ be the category of pairs (A, ψ), where A is an Artinian local Λ-algebra and $\psi : A/\mathfrak{m}_A \xrightarrow{\sim} k$ is an isomorphism of Λ-algebras. A morphism $(A, \psi) \to (A', \psi')$ in $\mathcal{C}_\Lambda$ is a local Λ-algebra homomorphism $f : A \to A'$ such that $\psi' \circ (f \mod \mathfrak{m}) = \psi$. The category $\mathcal{C}_\Lambda$ has a final object, given by (k, id_k). Note that if $\operatorname{Spec} k \to \operatorname{Spec}\Lambda' \subset \mathbb{S}$ is another factorization, the categories $\mathcal{C}_\Lambda, \mathcal{C}_{\Lambda'}$ are canonically isomorphic. Let

$$p : \mathcal{M} \to (\underline{\mathbf{Sch}}/\mathbb{S})$$

be an algebraic stack over $\mathbb{S}$, and let $x_0 : \operatorname{Spec} k \to \mathcal{M}$ be a morphism. By the 2-Yoneda lemma, we will identify x_0 with the object it defines in $\mathcal{M}(\operatorname{Spec} k)$ [Stacks, 04SS].

A *deformation* of x_0 over $(A,\psi) \in \mathcal{C}_\Lambda$ is by definition a pair (x,φ), where $x \in \mathcal{M}(\operatorname{Spec} A)$, and $\varphi : x_0 \to x$ is a morphism in $\mathcal{M}$ such that $p(\varphi) : \operatorname{Spec} k \to \operatorname{Spec} A$ induces the isomorphism $\psi : A/\mathfrak{m}_A \xrightarrow{\sim} k$. A morphism of deformations $(x,\varphi) \to (x',\varphi')$ is a map $f : x \to x'$ with $f \circ \varphi = \varphi'$. A morphism $f : (x,\varphi) \to (x',\varphi')$ of deformations over $(A,\psi) \in \mathcal{C}_\Lambda$ is an *A-isomorphism* if f is an isomorphism and $p(f) = \mathrm{id}_A$.

Definition 6.5.3. The *deformation functor* for x_0 is the functor

$$\begin{aligned} F_{x_0} &: \mathcal{C}_\Lambda \longrightarrow \underline{\textbf{Sets}}, \\ &(A,\psi) \mapsto \{\text{deformations of } x_0 \text{ over } (A,\psi)\}/A\text{-isomorphisms}. \end{aligned}$$

Remark 6.5.4. In a Deligne-Mumford stack, the diagonal (and hence inertia stack) is unramified (and hence formally unramified). This implies that a deformation over (A,ψ) has no non-trivial A-automorphisms.

If $f : \mathcal{M} \to \mathcal{N}$ is a morphism of algebraic stacks, then if (x,φ) is a deformation of x_0 over (A,ψ), then $(f(x), f(\varphi))$ is a deformation of $f(x_0)$ over (A,ψ). This defines a morphism of functors $f_* : F_{x_0} \to F_{f\circ x_0}$.

Let $\widehat{\mathcal{C}}_\Lambda$ be the category of pairs (R,ψ), where R is a Noetherian complete local Λ-algebra and $\psi : R/\mathfrak{m}_R \xrightarrow{\sim} k$ is a Λ-algebra isomorphism. Morphisms are local homomorphisms respecting ψ's. Thus, $\mathcal{C}_\Lambda$ embeds as a full subcategory of $\widehat{\mathcal{C}_\Lambda}$. Given $(R,\psi) \in \widehat{\mathcal{C}}_\Lambda$, it gives rise to a functor $h_{R,\psi} : \mathcal{C}_\Lambda \to \underline{\textbf{Sets}}$ defined by $h_R((A,\psi')) = \operatorname{Hom}_{\widehat{\mathcal{C}}_\Lambda}((R,\psi),(A,\psi'))$. A functor $F : \mathcal{C}_\Lambda \to \underline{\textbf{Sets}}$ is *pro-representable* by $(R,\psi) \in \widehat{\mathcal{C}}_\Lambda$ if there is an isomorphism $F \cong h_{R,\psi}$. If F_{x_0} is pro-represented by (R,ψ), then we say that R is a *universal deformation ring for x_0*.

Proposition 6.5.5. *Let $\mathcal{M}$ be a Deligne-Mumford stack with representable diagonal over a scheme $\mathbb{S}$. Let $\operatorname{Spec}\Lambda \subset \mathbb{S}$ a Noetherian open affine, and let k be a Λ-algebra which is a field. Let $\pi : U \to \mathcal{M}$ be a formally étale morphism with U a Noetherian scheme, let $\tilde{x}_0 : \operatorname{Spec} k \to U$ be a point with image $u \in U$, and let $x_0 := \pi \circ \tilde{x}_0$, with associated deformation functor $F_{x_0} : \mathcal{C}_\Lambda \to \underline{\textbf{Sets}}$ as above. Let $R := \mathcal{O}_{U,u}$. Assume that $\tilde{x}_0$ induces an isomorphism $\xi : R/\mathfrak{m}_R \xrightarrow{\sim} k$. Let $\widehat{R} := \lim_n R/\mathfrak{m}_R^n$ be the completion, with induced map $\widehat{\xi} : \widehat{R}/\mathfrak{m}_{\widehat{R}} \xrightarrow{\sim} k$. Then*

(a) *The map $\operatorname{Spec}\widehat{R} \to \mathcal{M}$ induces an isomorphism $\eta_{\widehat{R}} : h_{\widehat{R},\widehat{\xi}} \xrightarrow{\sim} F_{x_0}$.*

(b) *Let $z_0 : \operatorname{Spec}\Omega \to \mathcal{M}$ be a point with Ω separably closed, let $\kappa(z_0) \subset \Omega$ be the residue field of $\mathcal{O}_{\mathcal{M},z_0}$, and let $z_0' : \operatorname{Spec}\kappa(z_0) \to \mathcal{M}$ be the corresponding map. Then the map $\operatorname{Spec}\mathcal{O}_{\mathcal{M},z_0} \to \mathcal{M}$ identifies $\widehat{\mathcal{O}_{\mathcal{M},z_0}}$ with the universal deformation ring of z_0'.*

(c) *Let $\rho : V \to \mathcal{N}$ be a formally étale morphism from a Noetherian scheme V to a Deligne-Mumford stack $\mathcal{N}$ with representable diagonal, and suppose* *we are given a map $g : U \to V$ fitting into a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "g"] & \mathcal{M} \arrow[d, "f"] \\ V \arrow[r, "\rho"] & \mathcal{N}. \end{tikzcd}$$

Let $v := g(u)$, and suppose g induces an isomorphism of residue fields $\kappa(u) \xrightarrow{\sim} \kappa(v)$. Let $S := \mathcal{O}_{V,v}$, so g induces a map $S \to R$, which induces an isomorphism $\zeta : S/\mathfrak{m}_S \xrightarrow{\sim} k$. Then the corresponding diagram

$$\begin{tikzcd} h_{\widehat{R},\widehat{\xi}} \arrow[r, "\eta_{\widehat{R}}"] \arrow[d] & F_{x_0} \arrow[d, "f_*"] \\ h_{\widehat{S},\widehat{\zeta}} \arrow[r, "\eta_{\widehat{S}}"] & F_{f\circ x_0} \end{tikzcd}$$

commutes, where $\eta_{\widehat{R}}, \eta_{\widehat{S}}$ are as given in (a) *and $h_{\widehat{R},\widehat{\xi}} \to h_{\widehat{S},\widehat{\zeta}}$ is induced by g.*

Proof. Any map $a : (\widehat{R}, \widehat{\xi}) \to (A, \psi)$ with $(A, \psi) \in \mathcal{C}_\Lambda$ defines a map $x_a : \operatorname{Spec} A \to \operatorname{Spec} R \to \mathcal{M}$. Since $\mathcal{M}$ is fibered in groupoids, there is a map $\varphi : x_0 \to x_a$ in $\mathcal{M}$ inducing the isomorphism $\psi : A/\mathfrak{m}_A \xrightarrow{\sim} k$, and this map φ is unique up to precomposing with A-isomorphisms. The pair (x_a, φ) is thus a deformation of x_0 over A, and this defines a morphism of functors

$$\eta_{\widehat{R}} : h_{\widehat{R},\widehat{\xi}} \to F_{x_0}.$$

Let $h_{R,\xi} : \mathcal{C}_\Lambda \to \underline{\mathbf{Sets}}$ be the functor sending (A, ψ) to the set of local Λ-algebra homomorphisms $a : R \to A$ satisfying $\psi \circ (a \mod \mathfrak{m}_R) = \xi$. Since any local homomorphism $\widehat{R} \to A$ must factor through $\widehat{R}/\mathfrak{m}_{\widehat{R}}^n \cong R/\mathfrak{m}_R^n$ for some n, the natural map $h_{R,\xi} \to h_{\widehat{R},\widehat{\xi}}$ is an isomorphism. Thus it suffices to show that the map

$$\eta_R : h_{R,\xi} \longrightarrow F_{x_0}$$

is an isomorphism. Let (x, φ) be a deformation of x_0 over $(A, \psi) \in \mathcal{C}_\Lambda$, so that $p(\varphi) : \operatorname{Spec} k \to \operatorname{Spec} A$ induces $\psi : A/\mathfrak{m}_A \xrightarrow{\sim} k$. The map $\varphi : x_0 \to x$ induces a unique isomorphism $\alpha_\varphi : x_0 \xrightarrow{\sim} x \circ p(\varphi)$. The triple $(\tilde{x}_0, p(\varphi), \alpha_\varphi)$ defines a map $t : \operatorname{Spec} k \to \operatorname{Spec} R \times_{\mathcal{M}} \operatorname{Spec} A$ making the following diagram commute (ignoring the dotted arrow for now):

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "t"] \arrow[d, "p(\varphi)"'] \arrow[rr, bend left, "\tilde{x}_0"] & \operatorname{Spec} R \times_{\mathcal{M}} \operatorname{Spec} A \arrow[r] \arrow[d, "\pi_{u,A}"] & \operatorname{Spec} R \arrow[d, "\pi_u"] \\ \operatorname{Spec} A \arrow[r, equal] \arrow[ur, dashed] & \operatorname{Spec} A \arrow[r, "x"] & \mathcal{M}. \end{tikzcd}$$

Since π is formally étale, the same is true of π_u and $\pi_{u,A}$ [Stacks, 04EG]. Thus, there is a unique dotted arrow making the diagram commute, and hence the corresponding map $\operatorname{Spec} A \to \operatorname{Spec} R$ gives an element of $h_{R,\xi}(A, \psi)$ inducing the deformation (x, φ). The existence of the dotted arrow implies that η_R is surjective, and the uniqueness implies that η_R is injective. This proves (a). Part (b) follows from (a) by setting $U = \operatorname{Spec} \mathcal{O}_{\mathcal{M},z_0}$, and (c) follows from the construction of the isomorphism $\eta_{\widehat{R}}$. □

Proposition 6.5.6. *Let $\mathcal{M}, \mathcal{N}$ be locally Noetherian Deligne-Mumford stacks with representable diagonal. Let $f : \mathcal{M} \to \mathcal{N}$ be a quasi-finite morphism. The following are equivalent*:

(a) *f is étale.*
(b) *For every point $z_0 : \operatorname{Spec} \Omega \to \mathcal{M}$ with Ω separably closed, the induced morphism of étale local rings $(f_{z_0})_* : \mathcal{O}_{\mathcal{N},f\circ z_0} \to \mathcal{O}_{\mathcal{M},z_0}$ is an isomorphism.*
(c) *For every point $z_0 : \operatorname{Spec} \Omega \to \mathcal{M}$ with Ω separably closed and $\kappa(z_0) = \Omega$ (cf. Definition 6.5.1), f induces an isomorphism of deformation functors $f_* : F_{z_0} \xrightarrow{\sim} F_{f\circ z_0}$.*

Proof. First we show (1) $\iff$ (2). Let $V \to \mathcal{N}$ be an étale neighborhood of $f \circ z_0$. Then $\mathcal{M}_V := \mathcal{M} \times_{\mathcal{N}} V \to \mathcal{M}$ is étale and $\mathcal{M}_V$ is also a Deligne-Mumford stack with representable diagonal. Let $u : U \to \mathcal{M}_V$ be an étale neighborhood of some lift of z_0 to $\mathcal{M}_V$. Possibly shrinking U, V, we may assume that u is quasi-finite. Let $\tilde{z_0} : \operatorname{Spec} \Omega \to U$ be a lift of z_0 to U. Then f is étale at z_0 if and only if $\mathcal{M}_V \to V$ is étale at $u(\tilde{z_0})$ if and only if the map $h : U \to V$ is étale at the image of $\tilde{z_0}$ [Stacks, 02KM]. Since h is quasi-finite, the induced map on local rings is of finite presentation, so h is étale at the image of $\tilde{z_0}$ if and only if the induced map of local rings is weakly étale [Stacks, 0CKP, 039L], but this is equivalent to h inducing an isomorphism on étale local rings at $\tilde{z_0}$ [Stacks, 094Z]. Since $U \to \mathcal{M}$ and $V \to \mathcal{N}$ are étale, this map of étale local rings is precisely the map $(f_{z_0})_*$, so (1) $\iff$ (2).

Proposition 6.5.5 shows that (2)⇒(3). The converse follows from [Liu02, §4.3, Prop. 3.26]. □

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