FAMILIES OF ELLIPTIC CURVES OVER THE FOUR-POINTED CONFIGURATION SPACE AND A SEQUENCE OF DYER-FORMANEK-GROSSMAN

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Abstract. We show that the configuration space of four unordered points in $\mathbb{C}$ is a torsor under the affine group $\mathrm{Aff} := \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{C})}(\infty)$ over the universal punctured elliptic curve $\mathcal{E}^\circ \to \mathcal{M}_{1,1}$. We use this to give a geometric interpretation of an exceptional relationship, due to Dyer–Formanek–Grossman, between the four-stranded braid group and the automorphism group of a rank 2 free group. We also show how this gives a new moduli interpretation for this configuration space in terms of data associated to elliptic curves.

1. Introduction

Let $\mathrm{Conf}_n = \mathrm{Conf}_n(\mathbb{C})$ denote the configuration space of n-tuples of distinct ordered points in $\mathbb{C}$, and let $\mathrm{Conf}_{[n]} = \mathrm{Conf}_{[4]}(\mathbb{C})$ denote the quotient under the free action of S_n, i.e. the space of configurations of n unordered distinct points. An unordered configuration $\tau = \{a, b, c, d\} \in \mathrm{Conf}_{[4]}(\mathbb{C})$ determines the smooth curve E_τ of genus 1 given by the expression

$$Y^2 = (X-a)(X-b)(X-c)(X-d). \tag{1}$$

However, since τ is unordered, there is no canonical way to endow E_τ with a basepoint and thus this does not yield a map from $\mathrm{Conf}_{[4]}(\mathbb{C})$ to $\mathcal{M}_{1,1}$, the moduli space of elliptic curves, let alone a map to the universal punctured elliptic curve $\mathcal{E}^\circ$ over $\mathcal{M}_{1,1}$. The first goal of this note is to explain how to produce a map $\mathrm{Conf}_{[4]}(\mathbb{C}) \to \mathcal{E}^\circ$ which is moreover a torsor for the affine group $\mathrm{Aff} := \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{C})}(\infty) \cong \mathbb{C} \rtimes \mathbb{C}^*$.

To state the result, let $V \leqslant S_4$ denote the subgroup of order 4 generated by the conjugacy class of (12)(34). Let $\mathcal{Y}(2)$ denote the moduli stack classifying elliptic curves equipped with a trivialization of its 2-torsion, let $\mathcal{E}(2)$ denote the pullback of $\mathcal{E} \to \mathcal{M}_{1,1}$ over $\mathcal{Y}(2)$, let $\mathcal{E}(2)^\circ \subset \mathcal{E}(2)$ be the complement of the zero section, and let $\mathcal{E}(2)^* \subset \mathcal{E}(2)^\circ$ denote the complement of the 2-torsion. We will prove the following theorem:

Theorem A. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Conf}_4(\mathbb{C}) \arrow[r] \arrow[d, "\xi"] & \mathrm{Conf}_4(\mathbb{C})/V \arrow[r] \arrow[d, "\xi/V"] & \mathrm{Conf}_{[4]}(\mathbb{C}) \arrow[d, "\xi/S_4"] \\
\mathcal{E}(2)^* \arrow[r, "{[2]}"] \arrow[dr] & \mathcal{E}(2)^\circ \arrow[r, "\mathfrak{f}"] \arrow[d] & \mathcal{E}^\circ \arrow[d] \\
 & \mathcal{Y}(2) \arrow[r, "\mathfrak{f}"] & \mathcal{M}_{1,1}
\end{tikzcd} \tag{2}$$

where

(a) *Each rectangle is cartesian,*

(b) *each of $\xi, \xi/V$, and ξ/S_4 is a torsor for the affine group* Aff*, and*

(c) *the maps labeled $\mathfrak{f}$ are the forgetful maps.*

Theorem A grew out of a desire to have an algebro-geometric interpretation of a result of Dyer–Formanek–Grossman [DFG82], which was used in [CLT23] to produce infinitely many finite simple characteristic quotients of the free group of rank 2. In [DFG82], they describe a subgroup F of the four-stranded braid group $B_4 = \pi_1(\mathrm{Conf}_{[4]}(\mathbb{C}))$ in terms of the standard generators of B_4 and verify by direct computation that it has the following properties:

(a) F is free of rank 2,

(b) F is normal in B_4, and

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(c) the conjugation action of B_4 on F yields an isomorphism

$$B_4/Z(B_4) \cong \mathrm{Aut}^+(F), \tag{3}$$

and a short exact sequence

$$1 \to F \to B_4/Z(B_4) \to \mathrm{Out}^+(F) \to 1; \tag{4}$$

here the + indicates that we take the index-2 subgroup of $\mathrm{Out}(F)$ (or $\mathrm{Aut}(F)$) for which the induced automorphism on $\mathbb{Z}^2 \cong H_1(F;\mathbb{Z})$ has positive determinant.

Since the long-exact sequence of orbifold fundamental groups for the fibration $\mathcal{E}^\circ \to \mathcal{M}_{1,1}$ identifies $\pi_1(\mathcal{E}^\circ)$ with $\mathrm{Aut}^+(F)$, where F is interpreted as the fundamental group of a fiber, it follows that:

Corollary B. *The Dyer–Formanek–Grossman sequence* (4) *and the isomorphism* (3) *are induced by the long exact sequence in (orbifold) homotopy groups for the* Aff*-torsor* $\mathrm{Conf}_{[4]}(\mathbb{C}) \to \mathcal{E}^\circ$. *Under this isomorphism,* $\mathrm{Out}^+(F) \cong \mathrm{SL}_2(\mathbb{Z})$ *is interpreted as* $\pi_1^{orb}(\mathcal{M}_{1,1})$*, and* F *is interpreted as* $\pi_1(E^\circ)$*, the fundamental group of a fiber of* $\mathcal{E}^\circ \to \mathcal{M}_{1,1}$.

The map $\xi : \mathrm{Conf}_4(\mathbb{C}) \to \mathcal{E}(2)^*$ in Theorem A can be described as follows. Recall that a map $\mathrm{Conf}_4(\mathbb{C}) \to \mathcal{E}(2)^*$ is equivalent to the data of a family of elliptic curves $\mathbf{E}$ over $\mathrm{Conf}_4(\mathbb{C})$ equipped with a trivialization of its 2-torsion and with an additional section which is not 2-torsion. The family $\mathbf{E}$ corresponding to ξ is none other than the one described by the equation (1). As we will see in Lemma 3.4, this family is uniquely determined (up to isomorphism) by the properties:

- Its 2-torsion $\mathbf{E}[2]$ is split (has trivial monodromy over $\mathrm{Conf}_4(\mathbb{C})$), and
- as a double cover of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times \mathrm{Conf}_4(\mathbb{C}) \to \mathrm{Conf}_4(\mathbb{C})$ given by $(X,Y) \mapsto X$, it is branched above the sections given by a, b, c, d and is split above (the constant section) ∞.

A cubic equation for $\mathbf{E}$ is given in (10). The particular trivialization of $\mathbf{E}[2]$ used to define ξ is given by the preimages of the sections a, b, c, d, where the preimage of d will be taken to be the zero. The non-2-torsion section will be taken to be either of the two preimages above ∞ – different choices lead to (uniquely) isomorphic data, and hence the same map to $\mathcal{E}(2)^*$.[1] Once we have defined ξ, it remains to show that it descends to give a map $\mathrm{Conf}_{[4]}(\mathbb{C}) \to \mathcal{E}^\circ$. This proceeds in two steps. First, in §3.3, we show that the action of $V \leqslant S_4$ descends to the translation action of $\mathcal{E}(2)[2]$ on $\mathcal{E}(2)^*$. It is here that we see how to associate an elliptic curve to an *unordered* configuration $\{a, b, c, d\}$. Namely, we consider the quotient of the genus 1 curve $E_{a,b,c,d}$ by its 2-torsion, where the quotient is taken relative to *any* choice of a, b, c, d as the origin. This quotient is well-defined, and is itself a genus 1 curve isomorphic to $E_{a,b,c,d}$, but is now equipped with a distinguished point given by the identity coset. In the diagram (2), this quotient is expressed as the doubling map [2]. Finally, in §3.4, we show that the action of $S_4/V \cong S_3$ descends to the natural action of $\mathrm{SL}_2(\mathbb{Z}/2)$ on the set of trivializations of $\mathcal{E}(2)[2]$.

The rightmost column in the diagram of Theorem A realizes $\mathrm{Conf}_{[4]}(\mathbb{C})$ as an Aff-torsor over $\mathcal{E}^\circ$; in particular, the fibers over non-orbifold points are principal homogeneous spaces under Aff. Our final main result refines this, showing how to interpret the fibers in terms of data associated to elliptic curves. To do so, we recall that the *Hodge bundle* $\Omega\mathcal{M}_{1,1}$ is the line bundle on $\mathcal{M}_{1,1}$ whose fiber over an elliptic curve $E \in \mathcal{M}_{1,1}$ is the space $H^0(E;\Omega^1_E)$ of holomorphic differential 1-forms on E; its pullback over $\mathcal{E}^\circ$ will be denoted $\Omega\mathcal{E}^\circ$. We let $\Omega^\circ\mathcal{E}^\circ$ denote the complement of the zero section in the total space of $\Omega\mathcal{E}^\circ$. Since automorphisms of elliptic curves act freely on the space of nonzero holomorphic differentials, $\Omega^\circ\mathcal{E}^\circ$ is an algebraic variety. The points of $\Omega^\circ\mathcal{E}^\circ$ correspond to isomorphism classes of triples (E, Z, ω), where E is an elliptic curve, Z a nonzero point on E, and ω a nonzero holomorphic differential on E.

Theorem C. *There is an isomorphism of varieties*

$$f : \mathrm{Conf}_{[4]}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C} \times \Omega^\circ\mathcal{E}^\circ$$

given, on $\tau = \{a, b, c, d\} \in \mathrm{Conf}_{[4]}(\mathbb{C})$*, by*

$$f(\tau) = \left(cm(\tau), (\xi/S_4)(\tau), \frac{dX}{Y}\right),$$

[1] Since we described $\mathcal{E}(2)^*$ as a stack, it would be more correct to say that isomorphic data lead to *isomorphic* maps to $\mathcal{E}(2)^*$. However, since the data classified by $\mathcal{E}(2)^*$ have no nontrivial automorphisms, $\mathcal{E}(2)^*$ is representable (isomorphic to an algebraic variety), and hence when viewed as an algebraic variety, maps to $\mathcal{E}(2)^*$ are by definition isomorphism classes of maps to the moduli stack $\mathcal{E}(2)^*$. This abuse of terminology will be employed repeatedly in this note.

where $cm(\{a,b,c,d\}) = \frac{1}{4}(a+b+c+d)$ *denotes the center of mass of the configuration* τ*, and* $\frac{dX}{Y}$ *is the indicated form on (the smooth compactification of) the curve* $Y^2 = (X-a)(X-b)(X-c)(X-d)$.

Theorem C thus gives a novel moduli interpretation of $\mathrm{Conf}_{[4]}(\mathbb{C})$ in terms of elliptic curves and associated data.

Context. The relationship between square-free polynomials and (hyper)elliptic curves is a classical topic in algebraic geometry. The connection between *cubic* polynomials and elliptic curves, by way of the Weierstrass form, is widely known. One of the underlying aims of this note is to further elucidate the situation for polynomials of degree $n = 4$.

Via Birman–Hilden's theory of the hyperelliptic mapping class group (cf. [BH71] or [FM12, Section 9.4.1]), one can obtain a topological description of Dyer–Formanek–Grossman's group F as follows. Birman-Hilden theory gives an isomorphism $B_4/Z(B_4) \cong \mathrm{PMod}(\Sigma_{1,2})$, where $\mathrm{PMod}(\Sigma_{1,2})$ denotes the mapping class group of a surface $\Sigma_{1,2}$ of genus 1 with 2 individually-distinguishable marked points. The group F then arises as the "point-pushing subgroup", i.e. the kernel of the forgetful map $\mathrm{PMod}(\Sigma_{1,2}) \to \mathrm{Mod}(\Sigma_{1,1})$. This latter group is well-known to be $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$. The results of this article can be viewed as giving a "physical realization" of this story in terms of maps between moduli stacks.

2. Configuration spaces and the cross ratio

In this short section, we recall some properties of the cross ratio that underlie our constructions.

For a space X and an integer $n \geq 1$, let $\mathrm{Conf}_n(X)$ denote the subspace of X^n corresponding to tuples of n *distinct* points. The natural right action of the symmetric group S_n on $\mathrm{Conf}_n(x)$ by permuting coordinates is free. Explicitly, for $x = (x_1, \ldots, x_n) \in \mathrm{Conf}_n(X)$ and $\sigma \in S_n$, define x^σ by $(x^\sigma)_i = x_{\sigma(i)}$.[2] Let $\mathrm{Conf}_{[n]}(X)$ denote the quotient $\mathrm{Conf}_n(X)/S_n$. If $X = \mathbb{C}$, we simply write $\mathrm{Conf}_n := \mathrm{Conf}_n(\mathbb{C})$ and $\mathrm{Conf}_{[n]} := \mathrm{Conf}_{[n]}(\mathbb{C})$.

When $n = 4$, we denote by V the subgroup of S_4 generated by involutions without fixed points. Explicitly,

$$V = \{(\ ), (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Let $\mathbb{P}^1$ denote the Riemann sphere, and define

$$\mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}. \tag{5}$$

Recall that $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by the rule $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] z := \frac{az+b}{cz+d}$. The subgroup of scalar matrices $\{\pm I\}$ acts trivially, and the induced action of $\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\pm I$ is sharply 3-transitive; this means that it acts freely and transitively on $\mathrm{Conf}_3(\mathbb{P}^1)$. For $x = (x_1, x_2, x_3, x_4) \in \mathrm{Conf}_4(\mathbb{P}^1)$, define the *cross ratio* map χ by

$$\begin{aligned} \chi : \mathrm{Conf}_4(\mathbb{P}^1) &\longrightarrow \mathbb{P}^1 \\ x &\mapsto \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)} \end{aligned}$$

Proposition 2.1. *The cross ratio* χ *satisfies the following properties.*

(a) It's image is $\mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}$.

(b) It is invariant under the diagonal action of $\mathrm{PSL}_2(\mathbb{C})$ *on* $\mathrm{Conf}_4(\mathbb{P}^1)$ *as well as the action of* V.

(c) Two tuples $x, x' \in \mathrm{Conf}_4(\mathbb{P}^1)$ *lie in the same* $\mathrm{PSL}_2(\mathbb{C})$*-orbit if and only if their cross ratios agree. In particular,* χ *is a* $\mathrm{PSL}_2(\mathbb{C})$*-torsor, and the action of* V *can be induced by elements of* $\mathrm{PSL}_2(\mathbb{C})$.

(d) $\chi(0, 1, \lambda, \infty) = \lambda$

Proof. Items (a)-(c) are well-known (cf. [Con78, Definition 3.7 ff.]), and (d) follows by a straightforward calculation. ∎

[2] Our convention for multiplication in S_n is to treat permutations as functions. Thus it acts on the *left* on the set $\{1, 2, \ldots, n\}$. For example, we have $(12)(23) = (123)$.

3. Proof of Theorem A

3.1. **Moduli of elliptic curves with level 2 structure.** Recall that an elliptic curve is a compact Riemann surface E of genus 1, equipped with a marked point $O \in E$, which will play the role of the zero for its group law. Algebro-geometrically, for any scheme T, an elliptic curve over T is a proper, smooth, and finitely presented morphism $f : E \to T$ with geometrically connected fibers of genus 1, equipped with a zero section $O : T \to E$. In this context, one should think of E/T as a family of elliptic curves over T.

An elliptic curve E admits a unique involution $[-1]$. The quotient $E/[\pm 1]$ is isomorphic to the Riemann sphere $\mathbb{P}^1$, and the quotient map $E \to E/[\pm 1]$ is a double cover ramified at exactly the 2-torsion points $E[2] \subset E$. Conversely, by the classification of covering spaces, for any subset $B \subset \mathbb{P}^1$ of cardinality 4, there exists a unique (up to isomorphism) double cover E_B of $\mathbb{P}^1$ with branch locus B. The Hurwitz formula implies that E_B has genus 1. If B is equipped with an ordering, for example, if B is the underlying set of $\tau = (a, b, c, d)$ for some $\tau \in \mathrm{Conf}_4(\mathbb{P}^1)$, then we will by default view E_τ as an elliptic curve whose origin is the unique preimage of d. In this case E_τ is also equipped with an ordering of its points of order 2, as well as the data of a degree 2 map to $\mathbb{P}^1$. An explicit (affine) model for E_τ is given by

$$E_\tau : Y^2 = (X - a)(X - b)(X - c)(X - d).$$

In this model, the involution $[-1]$ is given by $(X, Y) \mapsto (X, -Y)$, and the double covering map to $\mathbb{P}^1$ is given by $(X, Y) \mapsto X$.

Let $\mathcal{Y}(2)$ be the moduli stack classifying quadruples (E, P, Q, R), where E is an elliptic curve, and P, Q, R is a labelling of the 2-torsion points of E. There are exactly three points of order 2, so P, Q, R can be viewed as an ordering of these points. Such an ordering is called a *level-2 structure* on E, and we call the quadruple (E, P, Q, R) an *enhanced elliptic curve*. We note that the automorphism group of any enhanced elliptic curve is the group generated by $[-1]$, which we simply denote by $[\pm 1]$. For $\tau \in \mathrm{Conf}_4$, let $P_\tau, Q_\tau, R_\tau \in E_\tau$ denote the preimages of a, b, c respectively. To such a configuration τ, we associate the enhanced elliptic curve $(E_\tau, P_\tau, Q_\tau, R_\tau)$.

For an object $(E, P, Q, R) \in \mathcal{Y}(2)$, choose an isomorphism $E/[\pm 1] \cong \mathbb{P}^1$, and let a, b, c, d denote the images of P, Q, R, O in $\mathbb{P}^1$ under this isomorphism. Different isomorphisms will result in different quadruples, but their cross ratio remains invariant, and hence we obtain a map

$$\begin{array}{rcl} \mathfrak{c} : \mathcal{Y}(2) & \longrightarrow & \mathbb{P}^* \\ (E, P, Q, R) & \mapsto & \chi(a, b, c, d) \end{array}$$

This map is a homeomorphism, and identifies $\mathbb{P}^*$ with the coarse scheme of $\mathcal{Y}(2)$. The map $\mathfrak{c}$ can be thought of as having degree $\frac{1}{2}$. It is sometimes referred to as the *modular λ-invariant*.

Remark 3.1. One can equivalently define $\mathcal{Y}(2)$ as the moduli stack classifying triples (E, P, Q) where P, Q is a basis for $E[2]$. Since $E[2]$ is a group scheme, the existence of sections P, Q of order 2 implies that $E[2]$ is totally split, and hence given P, Q, we can simply declare R to be the remaining section of order 2. Note that the natural action of $\mathrm{SL}_2(\mathbb{Z}/2)$ on the set of bases is isomorphic to the action of the symmetric group S_3 on the set of orderings of $E[2] - \{O\}$.

3.2. **Mapping Conf_4 to $\mathcal{E}(2)^*$.** The moduli stack $\mathcal{Y}(2)$ admits a *universal elliptic curve* $f : \mathcal{E}(2) \to \mathcal{Y}(2)$. This $\mathcal{E}(2)$ is the moduli stack classifying quintuples (E, P, Q, R, Z), where (E, P, Q, R) is an enhanced elliptic curve, and $Z \in E$. Two such quintuples are isomorphic if there is an isomorphism of the elliptic curves respecting each of the extra points P, Q, R, Z. The map f is defined by forgetting the point Z. The automorphism group of an object $(E, P, Q, R, Z) \in \mathcal{E}(2)$ consists of the automorphisms of E which fixes each of P, Q, R, Z. Since the only automorphism of elliptic curves fixing its 2-torsion is $[\pm 1]$, the only orbifold points of $\mathcal{E}(2)$ are the 2-torsion points; i.e., the points corresponding to (E, P, Q, R, Z) where $Z \in E[2]$. Let $\mathcal{E}(2)^*$ denote the open complement of its 2-torsion points; this is the substack of $\mathcal{E}(2)$ consisting of quintuples as above where moreover Z is disjoint from any 2-torsion section. Its objects have no automorphisms, and hence it is an algebraic variety. In this section we define an Aff-torsor

$$\xi : \mathrm{Conf}_4 \to \mathcal{E}(2)^*$$

Let $\mathcal{M}_{0,n} \cong \mathrm{Conf}_n(\mathbb{P}^1)/\mathrm{PGL}_2(\mathbb{C})$ denote the moduli stack of smooth projective curves of genus 0 with n disjoint sections. Since the action of PGL_2 is sharply 3-transitive on $\mathbb{P}^1$, $\mathcal{M}_{0,3}$ is a point, and $\mathcal{M}_{0,n}$ is an algebraic variety for $n \geq 3$. For $n = 4$, up to the action of PGL_2, we can assume that an object of $\mathcal{M}_{0,4}$ is given by $\mathbb{P}^1$ with the first three sections given by $0, 1, \infty$. The fourth section must be disjoint from these three, and hence $\mathcal{M}_{0,4} \cong \mathbb{P}^*$.

An explicit isomorphism is given by the cross ratio $\chi : \mathcal{M}_{0,4} \to \mathbb{P}^*$. For $n = 5$, forgetting the fifth section maps the moduli stack $\mathcal{M}_{0,5}$ onto $\mathcal{M}_{0,4}$ with fibers 4-punctured $\mathbb{P}^1$'s.

There is a natural map $\Psi : \mathcal{E}(2)^* \to \mathcal{M}_{0,5}$ obtained by sending a quintuple (E, P, Q, R, Z) to the genus 0 curve $E/[\pm 1]$, marked by the images of P, Q, R, O, Z, where $O \in E$ is the identity. Recall that $\mathcal{E}(2)^*$ maps to $\mathbb{P}^*$ via $\mathcal{E}(2)^* \to \mathcal{Y}(2) \xrightarrow{\mathfrak{c}} \mathbb{P}^*$, where $\mathfrak{c}$ is the modular lambda invariant.

Theorem 3.2. *The map $\Psi : \mathcal{E}(2)^* \to \mathcal{M}_{0,5}$ described above is an* isomorphism *of schemes over $\mathbb{P}^*$.*

Remark 3.3. Since the fibers of $\mathcal{M}_{0,5} \to \mathcal{M}_{0,4} \cong \mathbb{P}^*$ are 4-punctured $\mathbb{P}^1$'s, this means that the family of punctured elliptic curves $\mathcal{E}(2)^*$ over $\mathcal{Y}(2)$ "physically" looks like a family of punctured $\mathbb{P}^1$'s. What's happening is that the points of $\mathcal{E}(2)^*$ are in bijection with isomorphism classes of quintuples, and (E, P, Q, R, Z) is isomorphic to $(E, P, Q, R, -Z)$, even though $Z \neq -Z$ as points on E. Seen another way, whereas the fibers of the map $\mathcal{E}(2)^* \to \mathcal{Y}(2)$ are elliptic curves, it is the fibers of $\mathcal{E}(2)^* \to \mathcal{Y}(2) \xrightarrow{\mathfrak{c}} \mathbb{P}^*$ that are $\mathbb{P}^1$'s. Recall that $\mathfrak{c}$ should be thought of as having degree $\frac{1}{2}$.

To prove Theorem 3.2, we will need the following lemma:

Lemma 3.4. *Let T be a connected scheme, and let $\sigma_1, \ldots, \sigma_5$ be disjoint sections of $\mathbb{P}^1_T \to T$. Then up to isomorphism, there is a unique smooth projective curve E/T of genus 1 which is a double cover of $\mathbb{P}^1_T$ only branched over $\sigma_1, \ldots, \sigma_4$ and is split above σ_5.*

We will typically give E/T the structure of an elliptic curve whose zero section is σ_4. The sections $\sigma_1, \sigma_2, \sigma_3$ then define a level-2 structure on E/T.

Proof. Fix a base (geometric) point $t \in T$, and let $\mathbb{P}^1_t$ be the fiber of $\mathbb{P}^1_T$ over t. Let B be the union of the images of the sections $\sigma_1, \ldots, \sigma_4 \subset \mathbb{P}^1_T$, and let $B_t := B \cap \mathbb{P}^1_t$. We have a split exact sequence of fundamental groups [GR71, §XIII, Prop 4.3]

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\mathbb{P}^1_t - B_t, \sigma_5(t)) \arrow[r] & \pi_1(\mathbb{P}^1_T - B, \sigma_5(t)) \arrow[r] & \pi_1(T, t) \arrow[l, bend right, "(\sigma_5)_*"'] \arrow[r] & 1 \end{tikzcd} \tag{6}$$

Let $\gamma_1, \ldots, \gamma_4$ be "meridional" generators of $\pi_1(\mathbb{P}^1_t - B_t, \sigma_5(t))$ winding around the points $\sigma_1(t), \ldots, \sigma_4(t)$ respectively. The double cover of $\mathbb{P}^1_t$ branched only above B_t corresponds to a homomorphism $\varphi : \pi_1(\mathbb{P}^1_t - B_t, \sigma_5(t)) \to \mathbb{Z}/2$ sending $\gamma_1, \ldots, \gamma_4$ each to 1. To prove the lemma, it suffices to show that φ admits a unique extension to a homomorphism $\tilde{\varphi} : \pi_1(\mathbb{P}^1_T - B, \sigma_5(t)) \longrightarrow \mathbb{Z}/2$ which is zero on the image of $\pi_1(T, t)$.

Since (6) is split, the 5-term exact sequence for (6) degenerates into an isomorphism

$$H^1(\mathbb{P}^1_T - B; \mathbb{Z}/2) \cong H^1(T; \mathbb{Z}/2) \times H^1(\mathbb{P}^1_t - B_t; \mathbb{Z}/2)^{\pi_1(T,t)}.$$

Since B is a union of four distinct sections, the monodromy action of $\pi_1(T, t)$ on $H^1(\mathbb{P}^1_t - B_t; \mathbb{Z}/2)$ is trivial. Thus the desired $\tilde{\varphi}$ corresponds to the element $(0, \varphi) \in H^1(\mathbb{P}^1_T - B; \mathbb{Z}/2)$. □

Remark 3.5. The condition that the elliptic curve E/T is *split* above σ_5 is crucial to the uniqueness. It follows from the proof that there is a bijection between homomorphisms $\varphi : \pi_1(T, t) \to \mathbb{Z}/2$ and isomorphism classes of elliptic curves E_φ/T branched over exactly the sections $\sigma_1, \ldots, \sigma_4$. The curves E_φ for nontrivial $\varphi \neq \underline{0}$ are quadratic twists of $E_{\underline{0}}$.

Proof of Theorem 3.2. The Riemann-Hurwitz formula implies that E/T has genus 1. It is easy to check that Ψ commutes with the maps to $\mathbb{P}^*$. It remains to show that Ψ is an equivalence of categories. For any isomorphism of elliptic curves $\alpha : E \xrightarrow{\sim} E'$, the only other isomorphism which descends to the same map $E/[\pm 1] \xrightarrow{\sim} E/[\pm 1]$ is $[-1] \circ \alpha$. Since the only fixed points of $[-1]$ are 2-torsion points and since Z is not 2-torsion, only one of $\alpha, [-1] \circ \alpha$ will respect the section Z. This shows that Ψ is faithful.

Next, the existence part of Lemma 3.4 shows that the Ψ is essentially surjective. Finally, if $(E, \ldots), (E', \ldots)$ are objects of $\mathcal{E}(2)^*$, and $\beta : \Psi(E, \ldots) \to \Psi(E', \ldots)$ is an isomorphism, then the uniqueness part of Lemma 3.4 implies that there must exist an isomorphism $\alpha : (E, \ldots) \cong (E', \ldots)$ lifting β. This shows that Ψ is full, and hence completes the proof. □

There is a natural map $j : \mathrm{Conf}_4 \to \mathcal{M}_{0,5}$ sending (a, b, c, d) to the $\mathrm{PGL}_2(\mathbb{C})$-orbit of (a, b, c, d, ∞). This is visibly a torsor under the affine group $\mathrm{Aff} := \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{C})}(\infty) \cong \mathbb{C} \rtimes \mathbb{C}^\times$. Composing with the isomorphism Ψ^{-1} above, we have a map

$$\xi : \mathrm{Conf}_4 \xrightarrow{j} \mathcal{M}_{0,5} \xrightarrow{\Psi^{-1}} \mathcal{E}(2)^* \tag{7}$$

The pointed enhanced elliptic curve over Conf_4 corresponding to ξ can be described as follows. Let $\mathbf{P} := \mathrm{Conf}_4 \times \mathbb{P}^1$ be the trivial $\mathbb{P}^1$-bundle over Conf_4. Then by Lemma 3.4, there is up to isomorphism a unique elliptic double cover $\mathbf{E} \to \mathbf{P}$ ramified above a, b, c, d and split over ∞. We give $\mathbf{E}$ the structure of an elliptic curve by taking the unique section $O_{\mathbf{E}} : \mathrm{Conf}_4 \to \mathbf{E}$ above d to be the zero for the group law. Let $P_{\mathbf{E}}, Q_{\mathbf{E}}, R_{\mathbf{E}} : \mathrm{Conf}_4 \to \mathbf{E}$ denote the unique sections above a, b, c respectively; this defines a level 2 structure on $\mathbf{E}$. Finally, we once and for all choose a section $Z_{\mathbf{E}} : \mathrm{Conf}_4 \to \mathbf{E}$ above ∞. Then the map $\xi : \mathrm{Conf}_4 \to \mathcal{E}(2)^*$ corresponds to the object

$$(\mathbf{E}, P_{\mathbf{E}}, Q_{\mathbf{E}}, R_{\mathbf{E}}, Z_{\mathbf{E}}) \in \mathcal{E}(2)^* \tag{8}$$

We note that the other section above ∞ is $-Z_{\mathbf{E}}$; replacing $Z_{\mathbf{E}}$ with $-Z_{\mathbf{E}}$ leads to the same (strictly speaking, isomorphic) map $\xi : \mathrm{Conf}_4 \to \mathcal{E}(2)^*$. An explicit affine equation for $\mathbf{E}$ is simply:

$$\mathbf{E} : Y^2 = (X-a)(X-b)(X-c)(X-d) \tag{9}$$

This equation visibly has the desired branching behavior above $\mathbf{P}$. It follows from standard calculations (see, e.g., [Sil09, §II.2]), that the smooth compactification of (9) is split above $\infty \subset \mathbf{P}$. Thus, by the uniqueness part of Lemma 3.4, (9) is indeed an equation for $\mathbf{E}$.

Remark 3.6. A cubic equation for $\mathbf{E}$ can be given as follows. For $(a, b, c, d) \in \mathrm{Conf}_4$ with cross ratio $\lambda = \chi(a, b, c, d)$, there is a unique Möbius transformation γ_{abcd} sending $(0, 1, \lambda, \infty, \frac{d-b}{a-b}) \mapsto (a, b, c, d, \infty)$. Explicitly, γ_{abcd} is given by

$$\gamma_{abcd}(z) = \frac{dz + a\frac{b-d}{a-b}}{z + \frac{b-d}{a-b}}$$

For $(a, b, c, d) \in \mathrm{Conf}_4$ with cross ratio χ, define the quantity

$$\mu = \mu(a, b, c, d) := \left(\frac{d-b}{a-b}\right)\left(\frac{d-b}{a-b} - 1\right)\left(\frac{d-b}{a-b} - \chi\right)$$

We note that μ never vanishes for $(a, b, c, d) \in \mathrm{Conf}_4$. Let γ be the automorphism of $\mathbf{P}$ defined by γ_{abcd}. Then pulling $\mathbf{E} \to \mathbf{P}$ back by γ yields an elliptic curve branched over $0, 1, \chi, \infty$ and split above $\frac{d-b}{a-b}$. The pullback $\gamma^*\mathbf{E}$ admits an equation

$$\gamma^*\mathbf{E} : Y^2 = \mu X(X-1)(X-\chi) \tag{10}$$

Indeed, the elliptic curve defined by this equation is branched above $0, 1, \chi, \infty$ and is split above $\frac{d-b}{a-b}$, so Lemma 3.4 implies that(10) is a valid equation for $\gamma^*\mathbf{E}$. In this model, the level 2 structure is given by the sections above $0, 1, \chi$, and the section $\gamma^* Z_{\mathbf{E}}$ is here given by

$$(a, b, c, d) \mapsto \left(\frac{d-b}{a-b}, \mu\right)$$

3.3. **Descending to a map** $\mathrm{Conf}_4 / V \to \mathcal{E}(2)^\circ$**.** Recall that since the action of S_4 commutes with the diagonal action of Aff, the action of S_4 on Conf_4 induces an action on $\mathcal{E}(2)^* \cong \mathcal{M}_{0,5}$. The goal of this subsection and the next is to study how this action descends. This amounts to understanding the relationship between the object $(\mathbf{E}, P_{\mathbf{E}}, Q_{\mathbf{E}}, R_{\mathbf{E}}, Z_{\mathbf{E}})$ and its pullback by elements of S_4.

We begin with the action of $V \subset S_4$ on Conf_4. We examine the action of $\sigma = (12)(34) \in V$, the rest being similar.

A configuration $\tau = (a, b, c, d) \in \mathrm{Conf}_4$ is mapped to the object $((E_\tau, O_\tau), P_\tau, Q_\tau, R_\tau, Z_\tau)$, where we recall O_τ is the point above d, playing the role of zero, P_τ, Q_τ, R_τ are the points above a, b, c respectively, and Z_τ is a point above ∞, determined by our choice of $Z_{\mathbf{E}}$ in the previous section. It follows that (b, a, d, c) is mapped to $((E_\tau, R_\tau), Q_\tau, P_\tau, O_\tau, Z_\tau)$, which is (uniquely) isomorphic, via the translation $[+R_\tau]$, to $((E_\tau, O_\tau), P_\tau, Q_\tau, R_\tau, Z_\tau + R_\tau)$. The story for other elements of V being similar, we find that:

Proposition 3.7. *The action of V on Conf_4 descends to the action of $\mathcal{E}(2)[2]$ on $\mathcal{E}(2)^*$ by translation.*

It follows that we have a map $\mathrm{Conf}_4/V \to \mathcal{E}(2)^*/\mathcal{E}(2)[2]$. The latter quotient can be replaced by the image of $\mathcal{E}(2)^*$ under the (fiberwise) doubling map $[2] : \mathcal{E}(2) \to \mathcal{E}(2)$. This image is the complement of the zero section in $\mathcal{E}(2)$, denoted $\mathcal{E}(2)^\circ := \mathcal{E}(2) - O$. Thus we have a diagram

$$\begin{tikzcd} \mathrm{Conf}_4 \arrow[r] \arrow[d, "\xi"] & \mathrm{Conf}_4/V \arrow[d, "\xi/V"] \\ \mathcal{E}(2)^* \arrow[r, "{[2]}"] & \mathcal{E}(2)^\circ \end{tikzcd} \tag{11}$$

Proposition 3.8. *The diagram* (11) *is cartesian, and both vertical maps are* Aff*-torsors.*

Proof. There is a canonical map $\mathrm{Conf}_4 \to \mathcal{E}(2)^* \times_{\mathcal{E}(2)^\circ} (\mathrm{Conf}_4/V)$ of coverings of Conf_4/V. Since both covers are connected and of degree 4, this map is an isomorphism, and hence the diagram is cartesian. Since the actions of Aff and V on Conf_4 commute, the Aff-action on ξ descends to one on ξ/V. Since the property of being a torsor is local on the base and $[2]^*(\xi/V) = \xi$ is a torsor, ξ/V is one as well. □

3.4. **Descending to a map** $\mathrm{Conf}_{[4]} \to \mathcal{E}^\circ$**.** Let $\mathcal{E}$ be the universal elliptic curve over $\mathcal{M}_{1,1}$, and let $\mathcal{E}^\circ := \mathcal{E} - O$ be the complement of the zero section. We have a cartesian diagram

$$\begin{tikzcd} \mathcal{E}(2)^\circ \arrow[r, "\mathfrak{f}"] \arrow[d] & \mathcal{E}^\circ \arrow[d] \\ \mathcal{Y}(2) \arrow[r, "\mathfrak{f}"] & \mathcal{M}_{1,1} \end{tikzcd}$$

where the horizontal arrows labelled $\mathfrak{f}$ are given by forgetting the level-2 structure. In particular, $\mathcal{E}^\circ$ is the moduli stack of pairs (E, Z), where E is an elliptic curve and Z is a section disjoint from the zero section. The composition

$$h : \mathrm{Conf}_4 \xrightarrow{\xi} \mathcal{E}(2)^* \xrightarrow{[2]} \mathcal{E}(2)^\circ \xrightarrow{\mathfrak{f}} \mathcal{E}^\circ$$

corresponds to the pair $(\mathbf{E}, [2]Z_{\mathbf{E}})$, and for $\sigma \in S_4$, the composite $h \circ \sigma$ corresponds to the pair $(\sigma^*\mathbf{E}, \sigma^*[2]Z_{\mathbf{E}})$.

We wish to show that the map h descends to a map $\mathrm{Conf}_4/S_4 \longrightarrow \mathcal{E}^\circ$. This amounts to showing that for every $\sigma \in S_4$, there is an isomorphism $\varphi_\sigma : \sigma^*(\mathbf{E}, [2]Z_{\mathbf{E}}) \xrightarrow{\sim} (\mathbf{E}, [2]Z_{\mathbf{E}})$ that satisfies the cocycle condition:

$$\varphi_{\sigma\circ\sigma'} = \varphi_\sigma \circ \varphi_{\sigma'} \qquad \text{for all } \sigma, \sigma' \in S_4.$$

We will do this by showing that the isomorphism φ_σ is *unique*, and therefore automatically satisfies the cocycle condition. This is the content of the following lemma.

Lemma 3.9. *For any $\sigma \in S_4$, there is a unique isomorphism $\mathbf{E} \xrightarrow{\sim} \mathbf{E}$ lifting $\sigma : \mathrm{Conf}_4 \to \mathrm{Conf}_4$ which preserves the section $[2]Z_{\mathbf{E}}$.*

Proof. We note that since $[2]Z_{\mathbf{E}}$ is not generically 2-torsion, if an isomorphism exists, then it is unique. It follows from our results in §3.3 that such an isomorphism exists for $\sigma \in V$. Thus it suffices to show that such an isomorphism exists for representatives of the quotient group S_4/V. We will consider those $\sigma \in S_4$ which fixes the 4th element.

Viewing σ as an automorphism of Conf_4, it lifts to an automorphism $\tilde{\sigma}$ of $\mathbf{P}$ given by $(\tau, z) \mapsto (\tau^\sigma, z)$. This lift $\tilde{\sigma}$ permutes the sections $a, b, c : \mathrm{Conf}_4 \to \mathbf{P}$, and fixes the section d as well as the constant section ∞. Thus, by Lemma 3.4, $\tilde{\sigma}$ lifts to an isomorphism $\mathbf{E} \to \mathbf{E}$. There are two such lifts, which are distinguished by their action on the sections lying over $\infty \subset \mathbf{P}$. There is thus a unique isomorphism $\tilde{\tilde{\sigma}} : \mathbf{E} \to \mathbf{E}$ which fixes $Z_{\mathbf{E}}$. Since $\tilde{\sigma}$ fixes d, $\tilde{\tilde{\sigma}}$ is an isomorphism of elliptic curves (recall that the zero $O_{\mathbf{E}}$ was chosen to be the section above d). It follows that $\tilde{\tilde{\sigma}}$ also preserves $[2]Z_{\mathbf{E}}$. □

Theorem 3.10. *The map $\mathfrak{f} \circ \xi/V : \mathrm{Conf}_4/V \to \mathcal{E}^\circ$ factors through $\mathrm{Conf}_{[4]}$. Thus, we have a diagram*

$$\begin{tikzcd} \mathrm{Conf}_4 \arrow[r] \arrow[d, "\xi"] & \mathrm{Conf}_4/V \arrow[r] \arrow[d, "\xi/V"] & \mathrm{Conf}_{[4]} \arrow[d, "\xi/S_4"] \\ \mathcal{E}(2)^* \arrow[r, "{[2]}"] & \mathcal{E}(2)^\circ \arrow[r, "\mathfrak{f}"] & \mathcal{E}^\circ \end{tikzcd} \tag{12}$$

Moreover, all squares are cartesian, and all vertical maps are Aff*-torsors.*

Proof. Lemma 3.9 implies that $\mathfrak{f} \circ [2] \circ \xi$ factors through $\mathrm{Conf}_{[4]}$. The properties of the diagram follow from the same arguments as used in the proof of Proposition 3.8, noting that $\mathrm{Conf}_4 / V \to \mathrm{Conf}_{[4]}$ and $\mathcal{E}(2)^\circ \to \mathcal{E}^\circ$ are connected coverings of the same degree (namely, 6). □

This completes the proof of Theorem A.

4. Proof of Theorem C

Recall the setup: we have the space $\Omega^\circ \mathcal{E}^\circ$, the complement of the zero section in the Hodge bundle over $\mathcal{E}^\circ$, and we consider the map

$$\begin{aligned} f : \mathrm{Conf}_{[4]} &\to \mathbb{C} \times \Omega^\circ \mathcal{E}^\circ \\ \tau &\mapsto \left(cm(\tau), (\xi/S_4)(\tau), \frac{dX}{Y} \right). \end{aligned}$$

Our goal is to show that this is an isomorphism. We first address a preliminary matter.

Lemma 4.1. *Let $\tau = \{a, b, c, d\} \in \mathrm{Conf}_{[4]}$ be given. The differential $\frac{dX}{Y}$ on*

$$U_\tau = V(Y^2 - (X-a)(X-b)(X-c)(X-d)) \subset \mathbb{C}^2$$

extends to a nonzero holomorphic form on the smooth compactification E_τ of U_τ.

Proof. The theory of the Poincaré residue map (see, e.g. [GH94, p. 147]) implies that the restriction to U_τ of $\frac{dX}{Y}$ as a meromorphic 1-form on $\mathbb{C}^2$ is holomorphic and nowhere vanishing. It therefore extends to a differential ω on the smooth compactification E_τ with poles only possibly at the two added points $\pm\infty$. However, ω lies in the -1-eigenspace under the elliptic involution ι, and $\pm\infty$ are exchanged by ι, showing that the coefficients of the divisor of ω at $\pm\infty$ are equal. Since $\deg \mathrm{div}(\omega) = 0$ and $\mathrm{div}(\omega)$ is supported on $\pm\infty$, it follows that $\mathrm{div}(\omega) = 0$ as was to be shown. □

Proof of Theorem C. The first step will be to lift to the setting of ordered configurations. Recall from Section 3.2 that the space $\mathcal{E}(2)^*$ denotes the space of quintuples (E, P, Q, R, Z), where E is an elliptic curve, $P, Q, R \in E[2]$ is a level-2 structure on E, and $Z \in E \setminus E[2]$ is an additional point. Let us denote a general such quintuple as E^+. Similarly, for $\tau \in \mathrm{Conf}_4$, let $E_\tau^+ := (E_\tau, P_\tau, Q_\tau, R_\tau, Z_\tau)$, where P_τ, Q_τ, R_τ are as in §3.1 and Z_τ is determined by our choice of $Z_{\mathbf{E}}$ in §3.2. Recall that $\Omega^\circ \mathcal{E}(2)^*$ denotes the complement of the zero section in the total space of the Hodge bundle over $\mathcal{E}(2)^*$. Let $\tilde{f}$ be the map

$$\begin{array}{rcl} \tilde{f} : \mathrm{Conf}_4 & \longrightarrow & \mathbb{C} \times \Omega^\circ \mathcal{E}(2)^* \\ \tau & \mapsto & \left(cm(\tau), E_\tau^+, \frac{dX}{Y} \right). \end{array}$$

Thus we have a diagram

$$\begin{tikzcd} \mathrm{Conf}_4 \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{C} \times \Omega^\circ \mathcal{E}(2)^* \arrow[r] \arrow[d] & \mathcal{E}(2)^* \arrow[d] \\ \mathrm{Conf}_{[4]} \arrow[r, "f"] & \mathbb{C} \times \Omega^\circ \mathcal{E}^\circ \arrow[r] & \mathcal{E}^\circ. \end{tikzcd}$$

Note that Theorem A implies that the outer square is cartesian. Since the right square is cartesian, so is the left square. Moreover, the vertical maps are S_4-covers.

Theorem A shows that $\xi : \mathrm{Conf}_4 \longrightarrow \mathcal{E}(2)^*$ is an Aff-torsor. We will show that $\mathbb{C} \times \Omega^\circ \mathcal{E}(2)^* \to \mathcal{E}(2)^*$ is also an Aff-torsor, and that $\tilde{f}$ is a map of Aff-torsors, hence an isomorphism.

For $E^+ \in \mathcal{E}(2)^*$ and $A \in \mathrm{Aff}$ given by $A : z \mapsto \alpha z + \beta$ define

$$A \cdot (c, E^+, \omega) = (Ac, E^+, \alpha^{-1}\omega)$$

This visibly makes $\mathbb{C} \times \Omega^\circ \mathcal{E}(2)^*$ into an Aff-torsor over $\mathcal{E}(2)^*$. To show that $\tilde{f}$ is Aff-equivariant, we must show that $(E^+_{\alpha\tau+\beta}, \frac{dX}{Y})$ is isomorphic to $(E^+_\tau, \alpha^{-1}\frac{dX}{Y})$.

By construction, $\tau \in \mathrm{Conf}_4$ is assigned to the differential $\frac{dX}{Y}$ on E_τ, and $\alpha\tau+\beta$ is assigned to $\frac{dX}{Y}$ on the isomorphic curve $E_{\alpha\tau+\beta}$. There is a *unique* isomorphism $g : E_\tau \to E_{\alpha\tau+\beta}$ taking $E_\tau^+ \in \mathcal{E}(2)^*$ to $E^+_{\alpha\tau+\beta}$; we will see that $g^*\left(\frac{dX}{Y}\right) = \frac{1}{\alpha}\frac{dX}{Y}$. This reduces to a local calculation. One verifies that $g : E_\tau \to E_{\alpha\tau+\beta}$ is given on the affine part by

$$g(X,Y) = (\alpha X + \beta, \alpha^2 Y).$$

Thus

$$g^*\left(\frac{dX}{Y}\right) = \frac{\alpha\, dX}{\alpha^2 Y} = \frac{1}{\alpha}\frac{dX}{Y}.$$

Thus $\tilde{f}$ is an isomorphism of Aff-torsors. Taking quotients by the symmetric group S_4 shows that f is an isomorphism as desired. □

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